运筹学与控制论

DOI:10.3969/J. ISSN. 1672-6693. 2009. 04. 002

A New Extragradient Method for Variational Inequalities*

PENG Jian-wen

(College of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China)

Abstract: In this paper, we introduce a new approximation scheme based on the extragradient method and viscosity method for finding a common element of the set of solutions of the set of fixed points of a nonexpansive mapping and the set of the variational inequality for a monotone, Lipschitz continuous mapping. We obtain a strong convergence theorem for the sequences generated by these processes in Hilbert spaces as follows: Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let S be a nonexpansive mapping of C into H such that $Fix(S)\cap VI(C,A)\neq\varnothing$, where $Fix(S)$ and $VI(C,A)$, respectively, denote the set of fixed point of S and the solution set of a variational inequality. Let f be a contraction of H into itself and $\{x_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \gamma_n A x_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n) S P_C(x_n - \gamma_n A y_n) \end{cases}$$

for every $n=1,2,\cdots$, where $\{\gamma_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of numbers satisfying $\lim\limits_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$, $1>\limsup\limits_{n\to\infty}\beta_n\geqslant\liminf\limits_{n\to\infty}\beta_n>0$ and $\lim\limits_{n\to\infty}\gamma_n=0$. Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $w=P_{Fix(S)\cap VI(C,A)}f(w)$. The results in this paper improves some well-known results in the literature.

Key words: variational inequality; extragradient method; nonexpansive mapping; monotone mapping; viscosity method; strong convergence

中图分类号:O177.91 **文献标识码**:A **文章编号**:1672-6693(2009)04-0009-08

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H and let $P_C: H\to C$ be the metric projection of H onto C.

A mapping A of C into H is called monotone if

$$\langle Ax-Ay, x-y\rangle\geqslant 0$$

for all $x,y\in C$. A mapping A of C into H is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle\geqslant\alpha\|Ax-Ay\|^2$$

for all $x,y\in C$. A mapping $A: C\to H$ is called k-Lipschitz-continuous if there exists a positive real number k such that

$$\|Ax-Ay\|\leqslant k\|x-y\|$$

for all $x,y\in C$. It is easy to see that the class of α-inverse-strongly-monotone mappings does not contain some important classes of mappings even in a finite-dimensional case. For example, if the matrix in the corresponding linear complementarity problem is positively semidefinite, but not positively definite, then the mapping A will be monotone and Lipschitz-continuous, but not α-inverse-strongly-monotone.

Let the mapping $A: C\to H$ be monotone and k-Lipschitz-continuous. The variational inequality problem is

* **收稿日期**:2009-06-26 **修回日期**:2009-07-21
资助项目:国家自然科学基金(No. 10771228, No. 10831009) 重庆师范大学科研项目(No. 08XLZ05)
作者简介:彭健文,男,教授,博士,研究方向为最优化理论及其应用。

to find a $x \in C$ such that

$$\langle Ax, y - x \rangle \geqslant 0 \tag{1}$$

for all $y \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C,A)$.

Recall that a mapping S of a closed convex subset C of H is nonexpansive if there holds that $\|Sx - Sy\| \leqslant \|x - y\|$, for all $x, y \in C$. We denote the set of fixed points of S by $Fix(S)$.

It is known[1] that $F(S)$ is closed convex, but possibly empty. There are some methods for approximation of fixed points of a nonexpansive mapping. In 2000, Moudafi[2] introduced the viscosity approximation method for nonexpansive mappings (see [3] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n) Tx_n + \sigma_n f(x_n), n \geqslant 0 \tag{2}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved[2-3] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (2) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geqslant 0, x \in C \tag{3}$$

Takahashi and Toyoda[4] introduced the following iterative scheme for finding a common element of the set of solution of problem (1) and the set of fixed points of a nonexpansive mapping for an α-inverse strongly monotone mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \forall n \in \mathbf{N} \tag{4}$$

It is proved[4] that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (4) weakly converges to the unique solution x^* in $Fix(S) \cap VI(C,A)$.

On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean $\mathbf{R}^n$, Korpelevich[5] introduced the following so-called extragradient method

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda Ax_n) \\ x_{n+1} = P_C(x_n - \lambda Ay_n) \end{cases} \tag{5}$$

for every $n = 0, 1, 2, \cdots$, where $\lambda \in (0, \frac{1}{k})$. He showed that if $VI(C,A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (5), converge to the same point $z \in VI(C,A)$. The idea of the extragradient iterative process introduced by Korpelevich was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, e. g., the recent papers of He, Yang and Yuan[6], Gárciga Otero and Iuzem[7], Solodov and Svaiter[8], Solodov[9]. Moreover, Zeng and Yao[10] and Nadezhkina and Takahashi[11] introduced iterative processes based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for a monotone, Lipschitz continuous mapping. Yao and Yao[12] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for an α-inverse strongly monotone mapping.

In the present paper, we introduce a new viscosity approximation scheme based on the extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of the variational inequality for a monotone, Lipschitz continuous mapping. We obtain a strong convergence theorem for the sequences generated by these processes. The results in this paper improve some well-known results in the literature.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2$$

for all $x,y\in H$ and $\lambda\in[0,1]$.

For any $x\in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x-P_C(x)\|\leqslant\|x-y\|$ for all $y\in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C(x)\in C$ and

$$\langle x-P_C(x),P_C(x)-y\rangle\geqslant 0 \tag{6}$$

for all $x\in H$ and $y\in C$.

It is easy to see that (6) is equivalent to

$$\|x-y\|^2\geqslant\|x-P_C(x)\|^2+\|y-P_C(x)\|^2 \tag{7}$$

for all $x\in H$ and $y\in C$.

Let A be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (6) implies the following

$$u\in VI(C,A)\Rightarrow u=P_C(u-\lambda Au),\lambda>0$$

and

$$u=P_C(u-\lambda Au)\text{ for some }\lambda>0\Rightarrow u\in VI(C,A)$$

It is also known that H satisfies the Opial's condition[13], i. e., for any sequence $\{x_n\}\subset H$ with $x_n\rightharpoonup x$, the inequality $\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|$, holds for every $y\in H$ with $x\neq y$.

A set-valued mapping $T:H\rightarrow 2^H$ is called monotone if for all $x,y\in H$, $f\in Tx$ and $g\in Ty$ imply $\langle x-y,f-g\rangle\geqslant 0$. A monotone mapping $T:H\rightarrow 2^H$ is maximal if its graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f)\in H\times H$, $\langle x-y,f-g\rangle\geqslant 0$ for every $(y,g)\in G(T)$ implies $f\in Tx$. Let A be a monotone, k-Lipschitz continuous mapping of C into H and let N_Cv be normal cone to C at $v\in C$, i. e., $N_Cv=\{w\in H:\langle v-u,w\rangle\geqslant 0,\forall u\in C\}$. Define

$$Tv=\begin{cases}Av+N_Cv, & \text{if } v\in C\\ \varnothing, & \text{if } v\notin C\end{cases}$$

Then T is maximal monotone and $0\in Tv$ if and only if $v\in VI(C,A)$[14].

We will use the following results in the sequel.

Lemma 1[15-16] Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that $\alpha_{n+1}\leqslant(1-\gamma_n)\alpha_n+\delta_n$, where γ_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that 1) $\sum_{n=1}^{\infty}\gamma_n=\infty$; 2) $\limsup_{n\to\infty}\delta_n/\gamma_n\leqslant 0$ or $\sum_{n=1}^{\infty}|\delta_n|<\infty$; Then, $\lim_{n\to\infty}\alpha_n=0$.

Lemma 2 In a real Hilbert space H, there holds the following inequality $\|x+y\|^2\leqslant\|x\|^2+2\langle y,x+y\rangle$, for all $x,y\in H$.

Lemma 3[17] Let $\{x_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space, let $\{\beta_n\}$ be a sequence of real numbers such that $0<\liminf_{n\to\infty}\beta_n\leqslant\limsup_{n\to\infty}\beta_n<1$ for all $n=0,1,2,\cdots$ Suppose that $x_{n+1}=(1-\beta_n)w_n+\beta_nx_n$ for all $n=0,1,2,\cdots$ and $\limsup_{n\to\infty}\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|\leqslant 0$. Then, $\lim_{n\to\infty}\|w_n-x_n\|=0$.

3 Strong convergence theorems

In this section, we show a strong convergence of an iterative algorithm based on both viscosity approximation method and extragradient method which solves the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of the variational inequality for a monotone, Lipschitz continuous mapping in a Hilbert space.

Theorem 1 Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let S be a nonexpansive mapping of C into H such that $Fix(S) \cap VI(C,A) \neq \varnothing$. Let f be a contraction of H into itself and $\{x_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \gamma_n A x_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S P_C(x_n - \gamma_n A y_n) \end{cases}$$

for every $n = 1,2,\cdots$ where $\{\gamma_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of numbers satisfying the conditions

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty}\beta_n \geqslant \liminf_{n\to\infty}\beta_n > 0$;

(C3) $\lim_{n\to\infty}\gamma_n = 0$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $w = P_{Fix(S)\cap VI(C,A)} f(w)$.

Proof Let $\Omega = Fix(S) \cap VI(C,A)$. We show that $P_\Omega f$ is a contraction of C into itself. In fact, there exists $a \in [0,1)$ such that $\| f(x) - f(y) \| \leqslant a \| x - y \|$ for all $x, y \in C$. So, we have

$$\| P_\Omega f(x) - P_\Omega f(y) \| \leqslant \| f(x) - f(y) \| \leqslant a \| x - y \|$$

for all $x, y \in C$. Since H is complete, there exists a unique element $u_0 \in C$ such that $u_0 = P_\Omega f(u_0)$.

Let $t_n = P_C(x_n - \gamma_n A y_n)$ for every $n = 1,2,\cdots$ and $u \in \Omega$. Then $u = P_C(u - \gamma_n) A(u)$. From(7), the monotonicity of A, and $u \in VI(C,A)$, we have

$$\| t_n - u \|^2 \leqslant \| x_n - \gamma_n A y_n - u \|^2 - \| x_n - \gamma_n A y_n - t_n \|^2 =$$
$$\| x_n - u \|^2 - \| x_n - t_n \|^2 + 2\gamma_n \langle A y_n, u - t_n \rangle =$$
$$\| x_n - u \|^2 - \| x_n - t_n \|^2 + 2\gamma_n (\langle A y_n - Au, u - y_n \rangle + \langle Au, u - y_n \rangle + \langle A y_n, y_n - t_n \rangle) \leqslant$$
$$\| x_n - u \|^2 - \| x_n - t_n \|^2 + 2\gamma_n \langle A y_n, y_n - t_n \| \leqslant$$
$$\| x_n - u \|^2 - \| x_n - y_n \|^2 - 2\langle x_n - y_n, y_n - t_n \rangle - \| y_n - t_n \|^2 + 2\gamma_n \langle A y_n, y_n - t_n \rangle =$$
$$\| x_n - u \|^2 - \| x_n - y_n \|^2 - \| y_n - t_n \|^2 + 2\langle x_n - \gamma_n A y_n - y_n, t_n - y_n \rangle$$

Further, since $y_n = P_C(x_n - \gamma_n A x_n)$ and A is k-Lipschita-continuous, we have

$$\langle x_n - \gamma_n A y_n - y_n, t_n - y_n \rangle = \langle x_n - \gamma_n A x_n - y_n, t_n - y_n \rangle + \langle \gamma_n A x_n - \gamma_n A y_n, t_n - y_n \rangle \leqslant$$
$$\langle \gamma_n A x_n - \gamma_n A y_n, t_n - y_n \rangle \leqslant \gamma_n k \| x_n - y_n \| \| t_n - y_n \|$$

So, we have $\| t_n - u \|^2 \leqslant \| x_n - u \|^2 - \| x_n - y_n \|^2 - \| y_n - t_n \|^2 + 2\gamma_n k \| x_n - y_n \| \| t_n - y_n \| \leqslant$

$$\| x_n - u \|^2 - \| x_n - y_n \|^2 - \| y_n - t_n \|^2 + \gamma_n^2 k^2 \| x_n - y_n \|^2 + \| t_n - y_n \|^2 =$$
$$\| x_n - u \|^2 + (\gamma_n^2 k^2 - 1) \| x_n - y_n \|^2 \leqslant \| x_n - u \|^2 \tag{8}$$

Put $M_0 = \max\{\| x_1 - u \|, \frac{1}{1-a}\| f(u) - u \|\}$. It is obvious that $\| x_1 - u \| \leqslant M_0$. Suppose $\| x_n - u \| \leqslant M_0$. Then, from (8) and $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S t_n$, we have $u = Su$ and

$$\| x_{n+1} - u \| = \| \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S t_n - u \| \leqslant$$
$$\alpha_n \| f(x_n) - f(u) \| + \alpha_n \| f(u) - u \| + \beta_n \| x_n - u \| + (1 - \alpha_n - \beta_n) \| S t_n - u \| \leqslant$$
$$\alpha_n a \| x_n - u \| + \alpha_n \| f(u) - u \| + \beta_n \| x_n - u \| + (1 - \alpha_n - \beta_n) \| S t_n - u \| \leqslant$$
$$\alpha_n a \| x_n - u \| + \alpha_n \| f(u) - u \| + \beta_n \| x_n - u \| + (1 - \alpha_n - \beta_n) \| t_n - u \| \leqslant$$
$$\alpha_n a \| x_n - u \| + \alpha_n \| f(u) - u \| + \beta_n \| x_n - u \| + (1 - \alpha_n - \beta_n) \| x_n - u \| =$$
$$(1-a)\alpha_n \frac{\| f(u) - u \|}{1-a} + [1 - (1-a)\alpha_n] \| x_n - u \| \leqslant (1-a)\alpha_n M_0 + [1 - (1-a)\alpha_n) M_0 = M_0 \tag{9}$$

for every $n = 1,2,\cdots$ Therefore, $\{x_n\}$ is bounded. From (8), we also obtain that $\{t_n\}$ are bounded.

From $y_n = P_C(x_n - \gamma_n A x_n)$ and the monotonicity and the Lipschitz continuity of A, we have

$$\| y_n - u \|^2 = \| P_C(x_n - \gamma_n A x_n) - P_C(u - \gamma_n A u) \|^2 \leqslant \| x_n - \gamma_n A x_n - (u - \gamma_n A u) \|^2 =$$
$$\| x_n - u \|^2 - 2\gamma_n \langle A x_n - Au, x_n - u \rangle + \gamma_n^2 \| A x_n - Au \|^2 \leqslant$$

$$\| x_n - u \|^2 + \gamma_n^2 k^2 \| x_n - u \|^2 = (1 + \gamma_n^2 k^2) \| x_n - u \|^2$$

Hence, we obtain that $\{y_n\}$ is bounded. It follows from the Lipschitz continuity of A that $\{Ax_n\}$, $\{Ay_n\}$ are bounded. Since f and S are nonexpansive, we know that $\{f(x_n)\}$ and $\{St_n\}$ are also bounded. From the definition of t_n, we get

$$\begin{aligned} \| t_{n+1} - t_n \| &= \| P_C(x_{n+1} - \gamma_{n+1} Ay_{n+1}) - P_C(x_n - \gamma_n Ay_n) \| \leqslant \| (x_{n+1} - \gamma_{n+1} Ay_{n+1}) - (x_n - \gamma_n Ay_n) \| \leqslant \\ & \| (x_{n+1} - \gamma_{n+1} Ax_{n+1}) - (x_n - \gamma_{n+1} Ax_n) + \gamma_{n+1}(Ax_{n+1} - Ay_{n+1} - Ax_n) + \gamma_n Ay_n \| \leqslant \\ & \| x_{n+1} - x_n \| + \gamma_{n+1} \| Ax_{n+1} - Ax_n \| + \gamma_{n+1} \| Ax_{n+1} - Ay_{n+1} - Ax_n \| + \gamma_n \| Ay_n \| \leqslant \\ & \| x_{n+1} - x_n \| + k\gamma_{n+1} \| x_{n+1} - x_n \| + \gamma_{n+1} \| Ax_{n+1} - Ay_{n+1} - Ax_n \| + \gamma_n \| Ay_n \| \leqslant \\ & \| x_{n+1} - x_n \| + (\gamma_{n+1} + \lambda_n) M_1 \end{aligned} \tag{10}$$

where M_1 is an approximate constant such that

$$M_1 \geqslant \sup_{n \geqslant 1} \{ k \| x_{n+1} - x_n \| + \| Ax_{n+1} - Ay_{n+1} - Ax_n \| + \| Ay_n \| \}$$

Define a sequence $\{v_n\}$ such that $x_{n+1} = \beta_n x_n + (1 - \beta_n) v_n, \forall n \geqslant 1$

Then, we have

$$\begin{aligned} v_{n+1} - v_n &= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \\ & \frac{\alpha_{n+1} f(x_{n+1}) + (1 - \alpha_{n+1} - \beta_{n+1}) St_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + (1 - \alpha_n - \beta_n) St_n}{1 - \beta_n} = \\ & \frac{\alpha_{n+1}}{1 - \beta_{n+1}} f(x_{n+1}) - \frac{\alpha_n}{1 - \beta_n} f(x_n) + St_{n+1} - St_n + \frac{\alpha_n}{1 - \beta_n} St_n - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} St_{n+1} \end{aligned} \tag{11}$$

From (10) and (11), we have

$$\begin{aligned} & \| v_{n+1} - v_n \| - \| x_{n+1} - x_n \| \leqslant \\ & \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\| f(x_{n+1}) \| + \| St_{n+1} \|) + \frac{\alpha_n}{1 - \beta_n} (\| f(x_n) \| + \| St_n \|) + \| St_{n+1} - St_n \| - \| x_{n+1} - x_n \| \leqslant \\ & \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\| f(x_{n+1}) \| + \| St_{n+1} \|) + \frac{\alpha_n}{1 - \beta_n} (\| f(x_n) \| + \| St_n \|) + \| t_{n+1} - t_n \| - \| x_{n+1} - x_n \| \leqslant \\ & \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\| f(x_{n+1}) \| + \| St_{n+1} \|) + \frac{\alpha_n}{1 - \beta_n} (\| f(x_n) \| + \| St_n \|) + (\gamma_{n+1} + \gamma_n) M_1 \end{aligned}$$

It follows from (C1) ~ (C3) that $\limsup_{n \to \infty} (\| v_{n+1} - v_n \| - \| x_{n+1} - x_n \|) \leqslant 0$

Hence by Lemma 3, we have $\lim_{n \to \infty} \| v_n - x_n \| = 0$. Consequently

$$\lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} (1 - \beta_n) \| v_n - x_n \| = 0$$

Since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) St_n$, we have

$$\| x_n - St_n \| \leqslant \| x_{n+1} - x_n \| + \| x_{n+1} - St_n \| \leqslant \| x_{n+1} - x_n \| + \alpha_n \| f(x_n) - St_n \| + \beta_n \| x_n - St_n \|$$

and thus

$$\| x_n - St_n \| \leqslant \frac{1}{1 - \beta_n} (\| x_{n+1} - x_n \| + \alpha_n \| f(x_n) - St_n \|)$$

It follows from (C1) and (C2) that $\lim_{n \to \infty} \| x_n - St_n \| = 0$.

Since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) St_n$, for $u \in \Omega$, it follows from (8) that

$$\begin{aligned} \| x_{n+1} - u \|^2 &= \| \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) St_n - u \|^2 \leqslant \\ & \alpha_n \| f(x_n) - u \|^2 + \beta_n \| x_n - u \|^2 + (1 - \alpha_n - \beta_n) \| St_n - u \|^2 \leqslant \\ & \alpha_n \| f(x_n) - u \|^2 + \beta_n \| x_n - u \|^2 + (1 - \alpha_n - \beta_n) \| t_n - u \|^2 \leqslant \\ & \alpha_n \| f(x_n) - u \|^2 + \beta_n \| x_n - u \|^2 + (1 - \alpha_n - \beta_n) [\| x_n - u \|^2 + (\gamma_n{}^2 k^2 - 1) \| x_n - y_n \|^2] \leqslant \\ & \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n) \| x_n - u \|^2 + (1 - \alpha_n - \beta_n)(\gamma_n{}^2 k^2 - 1) \| x_n - y_n \|^2 \end{aligned} \tag{12}$$

from which it follows that

$$\begin{aligned} \| x_n - y_n \|^2 \leqslant & \frac{\alpha_n}{(1 - \alpha_n - \beta_n)(1 - \gamma_n{}^2 k^2)} (\| f(x_n) - u \|^2 - \| x_n - u \|^2) + \\ & \frac{1}{(1 - \alpha_n - \beta_n)(1 - \gamma_n{}^2 k^2)} (\| x_n - u \|^2 - \| x_{n+1} - u \|^2) \leqslant \end{aligned}$$

$$\frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}(\|f(x_n)-u\|^2-\|x_n-u\|^2)+$$
$$\frac{1}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}(\|x_n-u\|-\|x_{n+1}-u\|)\|x_{n+1}-x_n\| \tag{13}$$

It follows from (C1) ~ (C3) and $\|x_{n+1}-x_n\|\to 0$ that $\|x_n-y_n\|\to 0$.

By the same argument as in (8), we also have

$$\|t_n-u\|^2\leqslant\|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+2\gamma_nk\|x_n-y_n\|\|t_n-y_n\|\leqslant$$
$$\|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+\|x_n-y_n\|^2+\gamma_n{}^2k^2\|t_n-y_n\|^2=$$
$$\|x_n-u\|^2+(\gamma_n{}^2k^2-1)\|y_n-t_n\|^2$$

Combining the above inequality and (12), we have

$$\|x_{n+1}-u\|^2\leqslant\alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+(1-\alpha_n-\beta_n)\|t_n-u\|^2\leqslant$$
$$\alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+(1-\alpha_n-\beta_n)[\|x_n-u\|^2+(\gamma_n{}^2k^2-1)\|y_n-t_n\|^2]\leqslant$$
$$\alpha_n\|f(x_n)-u\|^2+(1-\alpha_n)\|x_n-u\|^2+(1-\alpha_n-\beta_n)(\gamma_n{}^2k^2-1)\|y_n-t_n\|^2$$

and thus

$$\|t_n-y_n\|^2\leqslant$$
$$\frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}(\|f(x_n)-u\|^2-\|x_n-u\|^2)+$$
$$\frac{1}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}(\|x_n-u\|^2-\|x_{n+1}-u\|^2)\leqslant$$
$$\frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}(\|f(x_n)-u\|^2-\|x_n-u\|^2)+$$
$$\frac{1}{\gamma_ny(1-\gamma_n{}^2k^2)}(\|x_n-u\|-\|x_{n+1}-u\|)\|x_{n+1}-x_n\|$$

which implies that $\|t_n-y_n\|\to 0$.

From $\|x_n-t_n\|\leqslant\|x_n-y_n\|+\|y_n-t_n\|$ we also have $\|x_n-t_n\|\to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n-At_n\|\to 0$.

Since

$$\|Sy_n-y_n\|\leqslant\|Sy_n-St_n\|+\|St_n-x_n\|+\|x_n-y_n\|\leqslant\|y_n-t_n\|+\|St_n-x_n\|+\|x_n-y_n\|$$

It follows that $\lim_{n\to\infty}\|Sy_n-y_n\|=0$.

Next we show that $\limsup_{n\to\infty}\langle f(u_0)-u_0,x_n-u_0\rangle\leqslant 0$, where $u_0=P_\Omega f(u_0)$. To show this inequality, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\langle f(u_0)-u_0,x_{n_j}-u_0\rangle=\limsup_{n\to\infty}\langle f(u_0)-u_0,x_n-u_0\rangle$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ which coverges weakly to w. Without loss of generality, we can assume that $\{x_{n_i}\}\rightharpoonup w$. From $\|x_n-x_n\|\to 0$, we obtain that $x_{n_i}\rightharpoonup w$. From $\|x_n-y_n\|\to 0$, we also obtain that $y_{n_i}\rightharpoonup w$. From $\|x_n-t_n\|\to 0$, we also obtain that $t_{n_i}\rightharpoonup w$. Since $\{x_{n_i}\}\subset C$ and C is closed and convex, we obtain $w\in C$.

We show that $w\in\Omega$. By similar argument with that in the proof of Theorem 1 in [11], it is easy to see that $w\in VI(C,A)$.

We next show that $w\in Fix(S)$. Assume $w\notin Fix(S)$. Since $y_{n_i}\rightharpoonup w$ and $w\neq Sw$, from the Opial condition we have

$$\liminf_{i\to\infty}\|y_{n_i}-w\|<\liminf_{i\to\infty}\|y_{n_i}-Sw\|=\liminf_{i\to\infty}\|y_{n_i}-Sy_{n_i}+Sy_{n_i}-Sw\|\leqslant$$
$$\liminf_{i\to\infty}\|y_{n_i}-Sy_{n_i}\|+\|Sy_{n_i}-Sw\|\leqslant\liminf_{i\to\infty}\|Sy_{n_i}-Sw\|\leqslant\liminf_{i\to\infty}\|y_{n_i}-w\|$$

which is a contradiction. So, we get $w\in Fix(S)$. This implies $w\in\Omega$. Therefore, we have

$$\limsup_{n\to\infty}\langle f(u_0)-u_0,x_n-u_0\rangle=\lim_{n\to\infty}\langle f(u_0)-u_0,x_{n_j}-u_0\rangle=\langle f(u_0)-u_0,w-u_0\rangle\leqslant 0 \tag{14}$$

Finally, we show that $x_n \to u_0$, where $u_0 = P_\Omega f(u_0)$.

From Lemma 2, we have

$$\begin{aligned}\|x_{n+1}-u_0\|^2 &= \|\alpha_n(f(x_n)-u_0)+\beta_n(x_n-u_0)+(1-\alpha_n-\beta_n)(St_n-u_0)\|^2 \leqslant \\ &\|\beta_n(x_n-u_0)+(1-\alpha_n-\beta_n)(St_n-u_0)\|^2+2\alpha_n\langle f(x_n)-u_0,x_{n+1}-u_0\rangle \leqslant \\ &(1-\alpha_n-\beta_n)\|St_n-u_0\|^2+\beta_n\|x_n-u_0\|^2+2\alpha_n\langle f(x_n)-u_0,x_{n+1}-u_0\rangle \leqslant \\ &(1-\alpha_n-\beta_n)\|St_n-u_0\|^2+\beta_n\|x_n-u_0\|^2+2\alpha_n\langle f(x_n)-f(u_0),x_{n+1}-u_0\rangle+2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle \leqslant \\ &(1-\alpha_n-\beta_n)\|t_n-u_0\|^2+\beta_n\|x_n-u_0\|^2+2\alpha_n a\|x_n-u_0\|\,\|x_{n+1}-u_0\|+2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle \leqslant \\ &(1-\alpha_n)\|x_n-u_0\|^2+\alpha_n a(\|x_n-u_0\|^2+\|x_{n+1}-u_0\|^2)+2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle\end{aligned}$$

and thus

$$\|x_{n+1}-u_0\|^2 \leqslant \left(1-\frac{\alpha_n}{1-a\alpha_n}\right)\|x_n-u_0\|^2+\frac{\alpha_n}{1-a\alpha_n}\langle 2f(u_0)-2u_0,x_{n+1}-u_0\rangle \tag{15}$$

It follows from Lemma 1, (14) and (15) that $\lim_{n\to\infty}\|x_n-u_0\| = 0$. From $\|y_n-x_n\|\to 0$, we have $y_n\to u_0$. The proof is now complete.

Corollary 1 Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let S be a nonexpansive mapping of C into H such that $Fix(S)\cap VI(C,A)\neq\varnothing$. Let v_0 be an arbitrary point in C, $\{x_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} x_1 = x\in C \\ y_n = P_C(x_n-\gamma_n Ax_n) \\ x_{n+1} = \alpha_n v_0+\beta_n x_n+(1-\alpha_n-\beta_n)SP_C(x_n-\gamma_n Ay_n)\end{cases}$$

for every $n = 1,2,\cdots$ where $\{\gamma_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of numbers satisfying the conditions

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty}\beta_n \geqslant \liminf_{n\to\infty}\beta_n > 0$;

(C3) $\lim_{n\to\infty}\gamma_n = 0$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $w = P_{Fix(S)\cap VI(C,A)}v_0$.

Proof Let $f(x) = v_0$ for all $x\in C$, by Theorem 1 we obtain the desired result.

Remark 1) Since the α-inverse strongly monotonicity of A has been weakened by the monotonicity and Lipschitz continuity of A. Theorem 1 and Corollary 1 improve Theorem 3.1 in[4] and Theorem 3.1 in[12].

2) Theorem 1 and Corollary 1 improve Theorem 3.1 in [10] by removeing the condition that $\lim_{n\to\infty}\|x_{n+1}-x_n\|\to 0$.

3) Theorem 1 and Corollary 1 improve and extend Theorem 3.1 in [4] and Theorem 3.1 in [11] from weak convergence to strong convergence.

References:

[1] Goebel K, Kirk W A. Topics on metric fixed-point theory [M]. Cambridge, England: Cambridge University Press, 1990.

[2] Moudafi A. Viscosity approximation methods for fixed-point problems[J]. J Math Anal Appl, 2000, 241: 46-55.

[3] Xu H K. Viscosity approximation methods for nonexpansive mappings[J]. J Math Anal Appl, 2004, 298: 279-291.

[4] Takahashi W, Toyoda M. Weak convergence theorems for nonexpansive mappings and monotone mappings[J]. J Optim Theory Appl, 2003, 118(2): 417-428.

[5] Korpelevich G M. The extragradient method for finding saddle points and other problems[J]. Matecon, 1976, 12: 747-756.

[6] He B S, Yang Z H, Yuan X M. An approximate proximal-extragradient type method for monotone variational inequalities [J]. J Math Anal Appl, 2004, 300: 362-374.

[7] Gárciga Otero R, Iuzem A. Proximal methods with penalization effects in Banach spaces[J]. Numer Funct Anal Optim, 2004, 25: 69-91.

[8] Solodov M V, Svaiter B F. An inexact hybrid generalized proximal point algorithm and some new results on the theory of Bregman functions[J]. Math Oper Res, 2000, 25: 214-230.

[9] Solodov M V. Convergence rate analysis of iteractive algo-

rithms for solving variatinal inequality problem[J]. Math Program,2003,96:513-528.

[10] Zeng L C,Yao J C. Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems[J]. Taiwan J Math,2006,10:1293-1303.

[11] Nadezhkina N, Takahashi W. Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings[J]. J Optim Theory Appl,2006,128:191-201.

[12] Yao Y H,Yao J C. On modified iterative method for nonexpansive mappings and monotone mappings[J]. Appl Math Comput,2007,186(2):1551-1558.

[13] Opial Z. Weak convergence of the sequence of successive approximation for nonexpansive mappings[J]. Bull Amer Math Soc,1967,73:561-597.

[14] Rockafellar R T. On the maximality of sums of nonlinear monotone operators[J]. Trans Amer Math Soc,1970,149:75-88.

[15] Xu H K. Iterative algorithms for nonlinear operators[J]. Journal of the London Mathematical Society,2002,66(1):240-256.

[16] Xu H K. An iterative approach to quadratic optimization [J]. Journal of Optimization Theory and Applications,2003,116:659-678.

[17] Suzuki T. Strong convergence theorems for an infinite family of nonexpansive mappings in general Banach spaces [J]. Fixed Point Theory and Applications,2005(1):103-123.

变分不等式的新的外梯度方法

彭 建 文

(重庆师范大学 数学与计算机科学学院,重庆 400047)

摘要 本文引入了一个新的求解非扩张映射的不动点集和具有单调及 Lipschitz 连续映射的变分不等式的解集的公共元素的近似算法。这一算法是建立在外梯度方法和粘性逼近方法基础上的。在 Hilbert 空间上得到了这一算法产生序列的强收敛性定理。其内容如下:设 C 是实 Hilbert 空间 H 中的非空闭凸集,映射 $A: C\to H$ 是单调和 k-Lipschitz 连续的,$S: C\to H$ 是非扩张映射满足 $Fix(S)\cap VI(C,A)\neq\varnothing$,其中 $Fix(S)$ 和 $VI(C,A)$ 分别是 S 的不动点集和变分不等式的解集,$f: H\to H$ 是压缩映射,序列 $\{x_n\}$ 和 $\{y_n\}$ 由下列算法产生的:$\begin{cases}x_1=x\in C\\ y_n=P_C(x_n-\gamma_nAx_n)\\ x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+(1-\alpha_n-\beta_n)SP_C(x_n-\gamma_nAy_n)\end{cases}$,$n=1,2,\cdots$,其中 $\{\gamma_n\}$,$\{\alpha_n\}$ 和 $\{\beta_n\}$ 是满足条件 $\lim\limits_{n\to\infty}\alpha_n=0$ 和 $\sum_{n=1}^{\infty}\alpha_n=\infty$,$1>\limsup\limits_{n\to\infty}\beta_n\geqslant\liminf\limits_{n\to\infty}\beta_n>0$ 和 $\lim\limits_{n\to\infty}\gamma_n=0$ 的数列,则 $\{x_n\}$ 和 $\{y_n\}$ 强收敛到 $w=P_{Fix(S)\cap VI(C,A)}f(w)$,这里 $P_{Fix(S)\cap VI(C,A)}f(w)$ 表示 $f(w)$ 在 $Fix(S)\cap VI(C,A)$ 上的投影。本文结果推广了文献中的一些著名结果。

关键词 变分不等式;外梯度方法;非扩张映射;单调映射;粘性逼近方法;收敛性定理

(责任编辑 黄 颖)