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ε-Efficiency in Vector Optimization Problems*

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Abstract: In this paper, a class of vector optimization problems is considered and ε-efficiency and two kinds of proper efficiency, namely ε-Benson proper efficiency and ε-Geoffrion proper efficiency are investigated. The equivalence is proved for two kinds of ε -proper efficiency. At the same time, ε-efficiency is characterized by making use of the classic scalarization method named as Benson's method which was introduced by Benson: x_0 is an ε-efficient solution of problem (VP) if and only if $\Psi=0$ for the scalar optimization problem (VPv) corresponds to (VP). Our results not only improve and generalize some known results and but also show that ε-proper efficiency introduced by Rong Weidong and Ma Yi coincides with ε-proper efficiency introduced by Liu Jen-chwan.

Key words: vector optimization; ε-efficiency; ε-proper efficiency; scalarization
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It is well known that the concepts of approximate solution has been playing an important role in vector optimization problems. Kutateladze initially introduced the concept of approximate solution named as ε-efficient solution and established vector variational principle, approximate Kuhn-Tucher conditions and approximate duality theorems in [1]. ε-Efficiency is an important kind of approximate efficiency in vector optimization problems and has been studied by some scholars in [2-4]. Recently, Liu proposed the concept of an ε-proper efficient solution by making use of the idea of Geoffrion proper efficiency and obtained some linear scalarization results in [5]. Rong and Ma proposed the concept of ε-proper efficiency in terms of the idea of Benson proper efficiency and established the linear scalarization theorems in [6].

Motivated by the works of [5-8], we prove the equivalence for two kinds of ε-proper efficiency introduced by Liu and Rong, respectively. Furthermore, we obtain some scalarization results of ε-efficiency by making use of the classic scalarization method named as Benson's method.

1 Preliminaries

In this section, we give some definitions and notations which will be used throughout this paper. Let $\mathbf{R}^n$ and $\mathbf{R}^m$ be n, m dimensional Euclidean space, respectively, $\mathbf{R}^m_+$ be the points in $\mathbf{R}^m$ with all coordinates positive or null and $\mathbf{R}^m_{++}$ be the points in $\mathbf{R}^m$ with all coordinates strictly positive. Analogous definitions for $\mathbf{R}^m_-$, $\mathbf{R}^m_{--}$. For any $x,y\in\mathbf{R}^m$, we consider the following inequalities. $x\geqq y\Leftrightarrow x_i\geqslant y_i$, for any $i=1,2,\cdots,m$; $x\geqslant y\Leftrightarrow x\geqq y$ and $x\neq y$; $x>y\Leftrightarrow x_i>y_i$, for any $i=1,2,\cdots,m$.

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The interior and closure of a set A are denoted by int A and $cl(A)$, respectively. The generated cone of a set A is defined as $cone(A)=\{\lambda a \mid \lambda \geqslant 0, a \in A\}$.

It is well known that $cone(A)$ is a convex cone if the set A is convex.

Consider the following vector optimization problem:

$$\text{(VP)} \quad \min \quad f(x)$$
$$\text{s.t.} \quad x \in S$$

where $S \subseteq \mathbf{R}^n$ and $f: S \to \mathbf{R}^m$. We assume that the feasible set S of (VP) is nonempty. Let $J=\{1,2,\cdots,m\}$.

Definition 1[1] Let $\varepsilon \in \mathbf{R}_+^m$. A point $x_0 \in S$ is said to be an ε-efficient solution of (VP) if there is no $x \in S$ such that $f(x) \leqslant f(x_0)-\varepsilon$. Denote $\varepsilon - E(f(S), \mathbf{R}_+^m)$ by the ε-efficient solution set of (VP).

Definition 2[5] A point $x_0 \in S$ is said to be an ε-proper efficient solution of (VP) if, ⅰ) x_0 is an ε-efficient solution of (VP); ⅱ) there exists a scalar $M>0$ such that for each i, we have $\frac{f_i(x_0)-f_i(x)-\varepsilon_i}{f_j(x)-f_j(x_0)+\varepsilon_j} \leqslant M$, for some j such that $f_j(x_0)<f_j(x)+\varepsilon_j$, whenever $x \in S$ and $f_i(x_0)>f_i(x)+\varepsilon_i$.

Denote $\varepsilon - PE(f(S), \mathbf{R}_+^m)$ by the ε-proper efficient solution set of (VP).

Definition 3[6] A point $x_0 \in S$ is said to be an ε-proper efficient solution of (VP) if

$$clcone(f(S)+\mathbf{R}_+^m+\varepsilon-f(x_0)) \cap (-\mathbf{R}_+^m)=\{0\}$$

2 Equivalence of ε-proper efficiency

In this section, we prove the equivalence of the definition of ε-proper efficiency introduced by Liu in [5] and the definition of ε-proper efficiency introduced by Rong and Ma in [6].

Theorem 1 Definition 2 is equivalent to Definition 3.

Proof Assume that x_0 satisfies Definition 3, it is clear that x_0 is an ε-efficient solution of (VP). Assume that ⅱ) is not true in Definition 2. Let M_k be an unbounded sequence of positive numbers. Without loss of generality, assume that for all M_k, there are $x^k \in S$ such that $f_1(x_0)>f_1(x^k)+\varepsilon_1$ and

$$\frac{f_1(x_0)-f_1(x^k)-\varepsilon_1}{f_j(x^k)-f_j(x_0)+\varepsilon_j}>M_k \tag{1}$$

for any $j \in \{2,3,\cdots,m\}$ with $f_j(x_0)<f_j(x^k)+\varepsilon_j$. Choosing a subsequence if necessary, we can assume that $\tilde{I}=\{i \in J \mid f_i(x^k)>f_i(x_0)-\varepsilon_i\}$ is constant for all k. Since x_0 is an ε-efficient solution of (VP), $\tilde{I}$ is a nonempty set. Let

$$t_k=\frac{1}{f_1(x_0)-f_1(x^k)-\varepsilon_1} \tag{2}$$

Clearly, $t_k>0$ for all k. Let $r_i^k=\begin{cases} 0, i=1 \\ 0, i \in \tilde{I} \\ f_i(x_0)-f_i(x^k)-\varepsilon_i, i \neq 1 \text{ and } i \notin \tilde{I} \end{cases}$. Clearly, $r^k \in \mathbf{R}_+^m$ for all k. From (1) and (2), we have $t_k\,(f_i(x^k)+r_i^k+\varepsilon_i-f_i(x_0))\begin{cases} =-1, i=1 \\ \in\left(0, \frac{1}{M_k}\right), i \in \tilde{I} \\ =0, i \neq 1 \text{ and } i \notin \tilde{I} \end{cases}$. The sequence converges to $d=(-1,0,0,\cdots,0)$ since $M_k \to \infty$. Obviously, $d \in clcone(f(S)+\mathbf{R}_+^m+\varepsilon-f(x_0)) \cap (-\mathbf{R}_+^m)$. There is a contradiction.

Conversely, assume that x_0 is an ε-efficient solution of (VP) and x_0 does not satisfy Definition 3.

Then there exists a nonzero vector d such that

$$d\in clcone(f(S)+\mathbf{R}_+^m+\varepsilon-f(x_0))\cap(-\mathbf{R}_+^m) \tag{3}$$

Without loss of generality, we may assume that $d_1<-1$ and $d_i\leqslant 0$ for $i=2,3,\cdots,m$. Hence from (3), there exist $\{t_k\}\subset\mathbf{R}_+$, $\{x^k\}\subset S$ and $\{r^k\}\subset\mathbf{R}_+^m$ such that

$$t_k(f(x^k)+r^k+\varepsilon-f(x_0))\to d \tag{4}$$

Choosing subsequences if necessary, we can assume that $\tilde{I}=\{i\in J\mid f_i(x^k)>f_i(x_0)-\varepsilon_i\}$ is the same for all k and nonempty by using ε-efficiency of x_0. Let $M>0$. From (4) and $t_k\to\infty$, we have that there exists k_0 such that for all $k\geqslant k_0$

$$f_1(x^k)-f_1(x_0)+\varepsilon_1<-\frac{1}{2t_k} \tag{5}$$

and

$$f_i(x^k)-f_i(x_0)+\varepsilon_i\leqslant\frac{1}{2Mt_k},i=2,3,\cdots,m \tag{6}$$

In particular, for $i\in\tilde{I}$ and $k\geqslant k_0$, it follows that from (6),

$$0<f_i(x^k)-f_i(x_0)+\varepsilon_i\leqslant\frac{1}{2Mt_k} \tag{7}$$

Hence, from (5) and (7), for $k\geqslant k_0$ and $i\in\tilde{I}$, we can obtain that $\dfrac{f_1(x_0)-f_1(x^k)-\varepsilon_1}{f_i(x^k)-f_i(x_0)+\varepsilon_i}>M$. There is a contradiction.

Remark 1 If $\varepsilon=0$, then Theorem 3.1 reduces to Theorem 3.2 in [7].

3 Scalarization and ε-efficiency

In this section, we obtain some scalarization results of ε-efficiency by making use of the classic scalarization method as Benson's method.

Consider the following scalar optimization problem corresponds to (VP):

$$(\mathrm{VP}_v)\qquad \Psi=\sup\sum_{j\in J}v_j,$$

$$\text{s. t.}\qquad \begin{cases} f_j(x_0)-v_j-\varepsilon_j-f_j(x)=0,\ \forall j\in J\\ v_j\geqslant 0,\ \forall j\in J\\ x\in S\end{cases}$$

Theorem 1 $x_0\in\varepsilon-E(f(S),\mathbf{R}_+^m)\Leftrightarrow\Psi=0$.

Proof Let (x,v) be a feasible solution of (VP_v). From $v_j\geqslant 0$ for $j\in J$ and the definition of v_j as $f_j(x_0)-\varepsilon_j-f_j(x)$, we have

$$\sum_{j\in J}v_j=0\Leftrightarrow v_j=0,j\in J\Leftrightarrow f_j(x_0)-\varepsilon_j-f_j(x)=0,j\in J \tag{8}$$

Assume that $x_0\notin\varepsilon-E(f(S),\mathbf{R}_+^m)$. Then there exists $\hat{x}\in S$ such that $f(\hat{x})\leqslant f(x_0)-\varepsilon$. This means that $v_j>0$ for some $j\in J$. But for $\Psi=0$ and (8), we know that it is impossible, i.e., $x_0\in\varepsilon-E(f(S),\mathbf{R}_+^m)$. On the other hand, if $x_0\in\varepsilon-E(f(S),\mathbf{R}_+^m)$, it is clear that $v_j=0$ for $j\in J$ and hence $\Psi=0$.

Theorem 2 If the supremum Ψ for (VP_v) is finite and is attained at the point (x_0,v^0), then $x_0\in\varepsilon-E(f(S),\mathbf{R}_+^m)$.

Proof Assume that $x_0\notin\varepsilon-E(f(S),\mathbf{R}_+^m)$. Then there exists $\hat{x}\in S$ such that $f(\hat{x})\leqslant f(x_0)-\varepsilon$ with at least one strict inequality. Define $\hat{v}=f(x_0)-f(\hat{x})-\varepsilon$. Then

$$\hat{v}_j=f_j(x_0)-f_j(\hat{x})-\varepsilon_j\geqslant f_j(x_0)-f_j(x_0)+\varepsilon_j=\varepsilon_j\geqslant 0,j\in J$$

Hence, $(\hat{x},\hat{v})$ is feasible for (VP_v) and $\hat{v}_j > v_j^0$ for some $j\in J$. Therefore, $\sum_{j\in J}\hat{v}_j > \sum_{j\in J} v_j^0$. This is impossible because (x_0, v^0) is an optimal solution of (VP_v).

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运筹学与控制论

向量优化问题的 ε-有效性

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摘要：研究了一类向量优化问题的 ε-有效性和两类真有效性，包括 ε-Benson 真有效性和 ε-Geoffrion 真有效性。首先证明了这两类真有效性之间的等价关系。同时，利用 Benson 标量化方法给出了向量优化问题的 ε-有效解的一些标量化结果。x_0 是问题(VP)的 ε-有效解当且仅当对应于问题(VP)的表量化问题(VP_v)有 $\Psi=0$。本文的主要结果不仅是对一些已有结果的改进与推广，而且也表明戎卫东与马毅提出的 ε-真有效性与 Liu Jen-chwan 提出的 ε-真有效性的一致性。

关键词：向量优化；ε-有效性；ε-真有效性；标量化

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