Operations Research and Cybernetics

DOI: CNKI: 50-1165/N. 20121112. 1642. 001

On Characterizing Solution Sets of Nonsmooth *B*-Preinvex Optimization Problems*

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Abstract: In this paper, various characterizations of optimal solution sets of nonsmooth B-preinvex optimization problems with inequality constrains are given. Firstly, making use of Clarke 's subdifferential, we establish the optimality condition for this kind of optimization problem; secondly, we presented a property about the solution set S of constrained B-preinvex optimization proble; finally, five equivalent characterizations of the solution set are obtained, that is, $S = \{x \in M \mid \langle \hat{\xi}, \eta(z,x) \rangle = 0, \exists \hat{\xi} \in \partial^c f(x) = (x \in M \mid \langle \hat{\xi}, \eta(z,x) \rangle \geqslant 0, \exists \hat{\xi} \in \partial^c f(x)\} = \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle = \langle \hat{\zeta}, \eta(z,x) \rangle, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\} = \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle \geqslant \langle \hat{\zeta}, \eta(z,x) \rangle, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\} = \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle = \langle \hat{\zeta}, \eta(z,x) \rangle = 0, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\}$. An example is given to illustrate that five solution sets are equal, i. e. $S = \{0\}$.

Key words: nonsmooth B-preinvex optimization; Clarke 's subdifferential; solution sets

中图分类号: O221.2 **文献标志码**: A **文章编号**: 1672-6693(2012)06-0001-05

It is well known that when an optimization problem has multiple optimal solutions, dual characterizations of the solution set are useful to characterize their boundedness and also for understanding the behavior of the solution methods. Mangasarian[1], Burke and Ferris[2] initially presented some excellent characterizations of the solution set for convex minimization problems over convex set when one solution is known. Since then, various extensions of these solution set characterizations to convex vector minimization problems, pseudolinear optimization programs and pseudoinvex extremum problems have been given in [3-7]. Some recent related study can be found about pre-invexity and the applications in optimization theory in [8-11].

The purpose of this article is to establish Lagrange multiplier characterizations of the solution set of the minimization of nonsmooth B-preinvex function subjected to explicit inequality constraints.

1 Prelininaries

Throughout this paper, let $X \subseteq \mathbf{R}^n$ be nonempty and denote by $\mathbf{R}^+$ the set of nonnegative real numbers. Suppose that $f: X \to \mathbf{R}$ and $g_i: X \to \mathbf{R}(i=1,2,\cdots,m)$ be locally Lipschitz functions. $\eta: X \times X \to \mathbf{R}$, and $b_i: X \times X \times [0,1] \to \mathbf{R}_+ (i=0,1,2,\cdots,m)$ such that $\lambda b_i(x,y,\lambda) \in [0,1]$ for all $x,y \in X$ and $\lambda \in [0,1]$.

In this section, we give some basic concepts and Lemmas which will be used in this paper.

Definition 1[12] A set $X \subseteq \mathbf{R}^n$ is said to be invex with respect to (in short, w. r. t.) η if there exists an $\eta: X \times X \to \mathbf{R}$ such that, for any $x, y \in X, \lambda \in [0,1], y + \lambda \eta(x,y) \in X$.

Definition 2[13] Let $X \subseteq \mathbf{R}^n$ be a nonempty invex set w. r. t. η. f is said to be B-preinvex on X w. r. t. η, b

* **Received**: 02-27-2012 **网络出版时间**: 2012-11-12 16:42:01

Foundation: The National Science Foundation of China (No. 11171363); the Special Fund of Chongqing Key Laboratory (No. CSTC 2011KLORSE02); the Education Committee Research Foundation of Chongqing (No. KJ110625)

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网络出版地址: http://www.cnki.net/kcms/detail/50.1165.N.20121112.1642.201206.1_001.html

if for any $x,y\in X,\lambda\in[0,1]$, $f(y+\lambda\eta(x,y))\leqslant\lambda b(x,y,\lambda)f(x)+(1-\lambda b(x,y,\lambda))f(y)$, $f:X\subseteq\mathbf{R}^n\to\mathbf{R}$ is said to be locally Lipschitz at a point $x\in X$ if there exists constant $K>0$ such that $|f(x)-f(y)|\leqslant K\|x-y\|$, for all x,y in a neighbourhood of x. We say that f is locally Lipschitz on X if it is locally Lipschitz at any point in X.

Let f be a locally Lipschitz at a given point $x\in X$. The Clarke 's[14] generalized directional derivative of f at $x\in X$ in the direction of a vector $v\in\mathbf{R}^n$, denoted by $f^0(x;v)$, is defined by

$$f^0(x;v)=\limsup_{y\to x,\lambda\downarrow 0}\frac{f(y+\lambda v)-f(y)}{\lambda}$$

and the Clarke 's[14] generalized subdifferential of f at $x\in X$, denoted by $\partial^c f(x)$, is defined by

$$\partial^c f(x)=\{\xi\in\mathbf{R}^n:f^0(x;v)\geqslant\langle\xi,v\rangle,\forall v\in\mathbf{R}^n\}$$

When f is locally Lipschitz at $x\in X$, f is said to be regular[14] at x if it is directionally differentiable at $x\in X$ and if $f^0(x;v)=f'(x;v)$, $\forall v\in\mathbf{R}^n$.

Lemma 1[8] Let f be a locally Lipschitz function on X. If f is B-preinvex w. r. t. η, b at $y\in X$, and $\lim_{\lambda\downarrow 0}b(x,y,\lambda)=b(x,y,0)$ for any $x,y\in X$. Furthermore, f is regular at y. Then $b(x,y,0)[f(x)-f(y)]\geqslant\langle\xi,\eta(x,y)\rangle$, $\forall\xi\in\partial^c f(y)$, $x,y\in X$.

In this paper, we consider the following Lipschitz B-preinvex optimization problem with inequality constraints:

$$(\mathrm{P})\begin{cases}\min f(x)\\ \text{s. t. } x\in D:=\{x\in X\,|\,g_i(x)\leqslant 0,i\in I=\{1,2,\cdots,m\}\}\end{cases}$$

where $X\subseteq\mathbf{R}^n$ is an invex set w. r. t. η, $f:X\to\mathbf{R}$ is a Lipschitz B-preinvex function w. r. t. η, b_0, $g_i:X\to\mathbf{R}(i\in I)$ are Lipschitz B-preinvex functions w. r. t. the same η and b_i, where $b_i(x,y,0)>0(i\in\{0\}\cup I)$ if $x\neq y$. Assume that the solution set of the problem (P), denoted by $S=\{x\in D\,|\,f(x)\leqslant f(y),\forall y\in D\}$ is nonempty. Let $x\in D$, $I(x)=\{i\in I\,|\,g_i(x)=0\}$ and $\tilde{I}(x)=\{i\in I(x)\,|\,\lambda_i>0\}$. It follows from [4] that the solution set S of the problem (P) is an invex set w. r. t. η.

For the problem (P), Clarke[14] and Long et al.[8] proved the following Karush Kuhn Tucker necessary and sufficient optimality condition, respectively.

Suppose that f is a B-preinvex w. r. t. η, b_i, $\lim_{\lambda\downarrow 0}b_i(x,y,\lambda)=b_i(x,y,0)(i\in I)$ for any $x,y\in X$, f and $g_i(i\in I)$ are regular at z. Furthermore, assume that some suitable constraint qualifications be satisfied. Then $z\in S$ if and only if there exists Lagrange multiplier $\mu=(\mu_1,\mu_2,\cdots,\mu_m)\in\mathbf{R}^m$ such that

$$0\in\partial^c f(z)+\sum_{i\in I}\mu_i\partial^c g_i(z) \tag{1}$$

$$\mu_i\geqslant 0,\lambda_i g_i(z)=0,\forall i\in I \tag{2}$$

2 Characterizations of the solution sets

In this section, we present the characterization of the solution set of the problem (P) in terms of Clarke 's subdifferentials and Lagrange multipliers.

Theorem 1 For the problem (P), let $z\in S$. Suppose that the optimality conditions (1) and (2) hold with a Lagrange multiplier $\mu=(\mu_1,\mu_2,\cdots,\mu_m)\in\mathbf{R}^m$, $\lim_{\lambda\downarrow 0}b_i(x,y,\lambda)=b_i(x,y,0)(i=1,2,\cdots,m)$ for any $x,y\in X$, f and $g_i(i\in I)$ are regular at z. Then $\sum_{i\in I(z)}\mu_i g_i(z)=0$ for all $x\in S$ and $f(\cdot)+\sum_{i\in I(z)}\mu_i g_i(\cdot)$ is constant on S.

Proof From (1), there exist $\hat{\xi}\in\partial^c f(z)$ and $\hat{\theta}\in\partial^c g_i(z)(i\in I)$ such that

$$\langle\hat{\xi},\eta(x,z)\rangle+\sum_{i\in I}\mu_i\langle\hat{\theta}_i,\eta(x,z)\rangle=0 \tag{3}$$

For each $x\in S$, then $f(x)=f(z)$. By the B-preinvexity of f w. r. t. η and Lemma 1, we obtain that, for any ξ

$\in \partial^c f(z)$

$$\langle \xi, \eta(x,z) \rangle \leqslant 0 \tag{4}$$

In follows from (3) and (4) that $\sum_{i \in I(z)} \mu_i \langle \hat{\theta}_i, \eta(x,z) \rangle \geqslant 0$. i. c.,

$$\sum_{i \in I(z)} \mu_i \langle \hat{\theta}_i, \eta(x,z) \rangle \geqslant 0 \tag{5}$$

By the B-preinvexity of g_i, $(i \in I(z))$ w. r. t. η and Lemma 1, it is clear that, for any $\eta_i \in \partial^c g_i(z)$, $b_i(x,z,0)[g_i(x) - g_i(z)] \geqslant \langle \eta_i, \eta(x,z) \rangle$. Then

$$b_i(x,z,0)\left[\sum_{i \in I(z)} \mu_i g_i(x) - \sum_{i \in I(z)} \mu_i g_i(z)\right] \geqslant \sum_{i \in I(z)} \mu_i \langle \theta_i, \eta(x,z) \rangle \tag{6}$$

From (2), (5) and (6), it follows that $\sum_{i \in I(z)} \mu_i g_i(x) \geqslant \sum_{i \in I(z)} \mu_i g_i(z) = 0$. By the feasibility of x, we can obtain that $\sum_{i \in I(z)} \mu_i g_i(x) \leqslant 0$. Hence, it is obvious that $\sum_{i \in I(z)} \mu_i g_i(x) = 0$. Consequently, $f(\cdot) + \sum_{i \in I(z)} \mu_i g_i(\cdot)$ is constant on S.

Theorem 2 For the problem (P), let $z \in S$. Suppose that the optimality conditions (1) and (2) hold with a Lagrange multiplier $\mu = (\mu_1, \mu_2, \cdots, \mu_m) \in \mathbf{R}^m$, $\lim_{\lambda \downarrow 0} b_i(x,y,\lambda) = b_i(x,y,0)$ $(i \in I)$ for any $x, y \in X$, f and g_i, $i \in I$ are regular at z. Denote $C(z) = \{\hat{\xi} \in \partial^c f(x) \mid \langle \hat{\xi}, \eta(x,z) \rangle \geqslant 0, \forall x \in D\}$, and $M = \{x \in X \mid g_i(x) = 0, \forall i \in \tilde{I}(z); g_i(x) \leqslant 0, \forall i \in I \backslash \tilde{I}(z)\}$.

Then $S = S_1 = S_2 = S_3 = S_4 = S_5$, where

$$S_1 := \{x \in M \mid \langle \hat{\xi}, \eta(z,x) \rangle = 0, \exists \hat{\xi} \in \partial^c f(x)\}$$

$$S_2 := \{x \in M \mid \langle \hat{\xi}, \eta(z,x) \rangle \geqslant 0, \exists \hat{\xi} \in \partial^c f(x)\}$$

$$S_3 := \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle = \langle \hat{\zeta}, \eta(z,x) \rangle, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\}$$

$$S_4 := \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle \geqslant \langle \hat{\zeta}, \eta(z,x) \rangle, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\}$$

$$S_5 := \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle = \langle \hat{\zeta}, \eta(z,x) \rangle = 0, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\}$$

Proof 1) $S \subseteq S_1$. Let $x \in S$. By theorem 1, we have $\sum_{i \in I(z)} \mu_i g_i(x) = 0$. From (2), it follows that $g_i(x) = 0$, $\forall i \in I \backslash \tilde{I}(z)$. That is, $x \in M$. From $x \in S$ and $z \in S$, we have $x + \lambda \eta(z,x) \in S$ for any $\lambda \in [0,1]$. Thus, $f(x + \lambda \eta(z,x)) = f(x)$, we have

$$\sup_{\|y-x\| < \delta, 0 < \lambda < \delta} \frac{f(y + \lambda \eta(z,x)) - f(y)}{\lambda} \geqslant \frac{f(y + \lambda \eta(z,x)) - f(x)}{\lambda} = 0$$

Thus, we can obtain that $f^0(x; \eta(z,x)) = \lim_{y \to x, \lambda \downarrow 0} \sup \frac{f(y + \lambda \eta(z,x)) - f(y)}{\lambda} \geqslant 0$.

Then there exists $\hat{\xi} \in \partial^c f(x)$, such that

$$\langle \hat{\xi}, \eta(z,x) \rangle \geqslant 0 \tag{7}$$

On the other hand, from $x \in S$ and $z \in S$, clearly, $f(x) = f(z)$. From the B-preinvexity of f w. r. t. η and Lemma 1, it follows that $\langle \xi, \eta(z,x) \rangle \leqslant 0$ for any $\xi \in \partial^c f(x)$. In particular,

$$\langle \hat{\xi}, \eta(z,x) \rangle \leqslant 0 \tag{8}$$

From (7) and (8), it follows that $\langle \hat{\xi}, \eta(z,x) \rangle = 0$. That is, $x \in S_1$. Thus $S \subseteq S_1$.

2) It is clear that $S_1 \subseteq S_2$.

3) $S_2 \subseteq S$. Assume that $x \in S_2$, we have $x \in M \subseteq D$ and

$$\langle \hat{\xi}, \eta(z,x) \rangle \geqslant 0, \exists \hat{\xi} \in \partial^c f(x) \tag{9}$$

By the B-preinvexity of f w. r. t. η, Lemma 1 and (9), we have $f(z) \geqslant f(x)$. From $z \in S$, it follows that f

$(z)=f(x)$. That is, $x\in S$. Thus $S_2\subseteq S$.

4) $S\subseteq S_3$. Let $x\in S$. From the proof of 1), we have $x\in M$ and there exists $\hat{\xi}\in\partial^c f(z)$, $\hat{\zeta}\in\partial^c f(x)$ such that $\langle\hat{\xi},\eta(x,z)\rangle=\langle\hat{\zeta},\eta(z,x)\rangle=0$. Clearly, $\hat{\xi}\in C(z)$. Thus $x\in S_3$ and we have $S\subseteq S_3$.

5) It is clear that $S_3\subseteq S_4$.

6) $S_4\subseteq S$. Assume that $x\in S_4$, we have $x\in M\subseteq D$ and

$$\langle\hat{\xi},\eta(x,z)\rangle\geqslant\langle\hat{\zeta},\eta(z,x)\rangle,\ \exists\hat{\xi}\in C(z),\hat{\zeta}\in\partial^c f(x) \tag{10}$$

Due to $\hat{\xi}\in C(z)$ and (10), we have $\langle\hat{\zeta},\eta(z,x)\rangle\geqslant 0$. By the B-preinvexity of f w. r. t. η and Lemma 1, we have $f(z)\geqslant f(x)$. From $z\in S$, it follows that $f(z)=f(x)$. That is, $x\in S$. Thus $S_4\subseteq S$.

7) $S\subseteq S_5$. From the proof of 4), it follows that $S\subseteq S_5$.

8) $S_5\subseteq S$. Clearly, $S_5\subseteq S_3$. From 5) and 4), it follows that $S_3\subseteq S_4\subseteq S$. Thus $S_5\subseteq S$.

The following example illustrate the above Theorem 2.

Example 1 Consider the following constrained B-preinvex optimization problem (P).

$$(\mathrm{P})\begin{cases}\min & f(x)=|x|\\ \text{s. t.} & g_1(x)=\dfrac{1}{2}x+\dfrac{3}{2}|x|-1\leqslant 0\\ & g_2(x)=-\dfrac{1}{2}x-\dfrac{1}{2}|x|\leqslant 0\\ & x\in X=\mathbf{R}\end{cases}$$

We can verify that f and $g_i\,(i=1,2)$ are Lipschitz with $L=2$. Moreover, let

$$\eta(x,y):=\begin{cases}x-y, x\geqslant 0, y\geqslant 0\\ x-y, x<0, y<0\\ -y, x\geqslant 0, y<0\\ -y, x<0, y\geqslant 0\end{cases}\quad b_0(x,y,\lambda):=\begin{cases}1, x\geqslant 0, y\geqslant 0\\ 1, x<0, y<0\\ 1-\lambda, x\geqslant 0, y<0\\ 1-\lambda^2, x<0, y\geqslant 0\end{cases}$$

$$b_1(x,y,\lambda):=\begin{cases}1, x\geqslant 0, y\geqslant 0\\ 1, x<0, y<0\\ 1-\lambda, x\geqslant 0, y<0\\ 1-\lambda, x<0, y\geqslant 0\end{cases}\quad b_2(x,y,\lambda):=\begin{cases}1, x\geqslant 0, y\geqslant 0\\ 1, x<0, y<0\\ 1-\lambda^2, x\geqslant 0, y<0\\ 1-\lambda, x<0, y\geqslant 0\end{cases}$$

Note that the set of feasible solutions for (P) is $D=[-1,0.5]$. It is not difficult to prove that all hypotheses of Theorem 2 are fulfilled. Clearly, $z=0\in S$ and $I(z)=\{2\}$. Moreover, let $\mu=(0,1)$. We can verify that the optimality conditions (1) and (2) hold with the Lagrange multiplier $\tilde{I}(z)=\{2\}$. We can verify that Theorem 2 are true. The solution set can be described as $S=S_1=S_2=S_3=S_4=S_5=\{0\}$.

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运筹学与控制论

非光滑 B-预不变凸优化问题的解集刻画

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摘要:给出了具有不等式约束的非光滑 B-预不变凸优化问题的最优解集的各种刻画。首先,利用 Clarke 次微分建立了该优化问题最优解的充分必要条件;再讨论了该优化问题在其解集 S 上的一个性质;最后建立了该优化问题解集的 5 种等价形式,即 $S = \{x \in M \mid \langle \hat{\xi}, \eta(z,x) \rangle = 0, \exists \hat{\xi} \in \partial^c f(x)\} = \{x \in M \mid \langle \hat{\xi}, \eta(z,x) \rangle \geqslant 0, \exists \hat{\xi} \in \partial^c f(x)\} = \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle = \langle \hat{\zeta}, \eta(z,x) \rangle, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\} = \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle \geqslant \langle \hat{\zeta}, \eta(z,x) \rangle, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\} = \{x \in M \mid \langle \hat{\xi}, \eta(x,z) \rangle = \langle \hat{\zeta}, \eta(z,x) \rangle = 0, \exists \hat{\xi} \in C(z), \hat{\zeta} \in \partial^c f(x)\}$,并举例验证这 5 个集合都相等,为 $S = \{0\}$。

关键词:非光滑 B-预不变凸优化;Clarke 次微分;解集

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