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# Outer Synchronization between Two Complex Dynamical Networks with Time-varying Delays*

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**Abstract**: In this paper, we study the outer synchronization between the drive network and the response network with time-varying delays. By employing linear matrix inequalities (LMI) and Lyapunov functional method, some new sufficient conditions ensuring the outer synchronization between two complex networks are obtained. The outer synchronization between the drive network and the response network is achieved if one of the following conditions is satisfied: 1) $0\leqslant\dot{\tau}(t)\leqslant\sigma<1$, $\boldsymbol{M}_i>0$, $\boldsymbol{S}_i>0$, $\begin{bmatrix}\boldsymbol{U}_i(t) & \lambda_i c\boldsymbol{M}_i\boldsymbol{\Gamma}\\ \lambda_i c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i & -(1-\sigma)\boldsymbol{S}_i\end{bmatrix}<0$, $i=2,\cdots,N$; 2) $\dot{\tau}(t)\leqslant 0$, $\tau(t)\leqslant\tau$, $0<\tau<\infty$, $\boldsymbol{M}_k>0$, $\boldsymbol{S}_k>0$, $\begin{bmatrix}\boldsymbol{U}_k & \lambda_k c\boldsymbol{M}_k\boldsymbol{\Gamma}-\boldsymbol{Y}_k & \tau\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Z}_k\\ \lambda_k c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_k-\boldsymbol{Y}_k^{\mathrm{T}} & -\boldsymbol{S}_k & \tau\lambda_k c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Z}_k\\ \tau\boldsymbol{Z}_k\boldsymbol{H} & \tau\lambda_k c\boldsymbol{Z}_k\boldsymbol{\Gamma} & -\tau\boldsymbol{Z}_k\end{bmatrix}<0$, $k=2,\cdots,N$. Finally, a numerical example is provided to illustrate the efficiency of the derived results.

**Key words**: complex networks; outer synchronization; time-varying delays; Lyapunov functional

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## Introduction

Over the past one decade, dynamical behaviors of complex networks have attracted a great deal of attention in variety of fields, such as communication networks, food webs, internet, World Wide Web, social networks, metabolic networks, power grid networks, biology, physics, mathematics, engineering and so on[1-5]. In particular, the synchronization is one of the most significant and interesting dynamical properties of the complex networks.

Recently, outer synchronization between two coupled complex dynamical networks have attracted more and more attention. Researches on outer synchronization of networks have the strong importance and potential applications in our life. For example, in order to know more about the communication of the infectious diseases, such as Mad Cows, AIDS and SARS, between animals and the human beings, it is required two different

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networks to distinguish animals from the human beings; investigating the interactions of protein network and gene network may disclose evolution process in systems biology[6]. These mean that to study the dynamics between two coupled networks is necessary and important. Li, Sun and Kurths firstly studied the synchronization between two complex networks which is called "outer synchronization"[7]. They studied the outer synchronization between two complex dynamical networks having the same coupling structure and the different coupling strength. Tang et al. investigated the theoretical analysis of synchronization between two complex networks with nonidentical topological structures[8]. By designing effective adaptive controllers, they achieve synchronization between two complex networks. Both the cases of identical and nonidentical network topological structures were considered and several useful criteria for synchronization were given. Wu et al. investigated the outer synchronization between two networks with different coupling structures and also provided the control law to achieve outer synchronization based on Barbalat's lemma[9]. Li et al. considered the synchronization between two discrete-time networks which have the same connection topologies and derived analytically a sufficient condition for achieving this outer synchronization[10]. On the basis of Lyapunov function approach, Li, et al. proved that for networks with balanced structure topology, outer synchronization can be asymptotically reached by using arbitrary coupling strength[11]. The synchronization problem of complex networks has been one of the focus points in many research and application fields. In addition, time delays commonly exist in various complex dynamical networks due to the finite information transmission and processing speeds among the network nodes, and some of time delays cannot be ignored.

Moreover, the delays are frequently varied with time and the elements of each node havethe same time-varying delays. Unfortunately, there are very few results developed in this direction.

Motivated by the above discussions, the objective of this article is to study the outer synchronization between two complex networks with time-varying coupling delays. Sufficient conditions ensuring the outer synchronization for complex networks associated with time-varying delays are obtained by LMI and Lyapunov functional method. The rest of this article is organized as follows. We study the drive-response complex dynamical network models and some useful preliminaries are given in Section 2. Several outer synchronization criteria are established in Section 3. In Section 4, a numerical example is given to illustrate the theoretical results.

# 1 Model and preliminaries

In this paper, we consider the driving network in the form

$$\dot{x}_i(t)=g(x_i(t))+c\sum\nolimits_{j=1}^{N}a_{ij}\boldsymbol{\Gamma}x_j(t-\tau(t)) \tag{1}$$

and the response network as

$$\dot{y}_i(t)=g(y_i(t))+\left(\boldsymbol{H}-\frac{\partial g(x_i)}{\partial x_i}\right)[y_i(t)-x_i(t)]+c\sum\nolimits_{j=1}^{N}a_{ij}\boldsymbol{\Gamma}y_j(t-\tau(t)) \tag{2}$$

where $i=1,2,\cdots,N$, $x_i(t)=(x_{i1}(t),x_{i2}(t),\cdots,x_{in}(t))^{\mathrm{T}}\in\mathbf{R}^n$, $y_i(t)=(y_{i1}(t),y_{i2}(t),\cdots,y_{in}(t))^{\mathrm{T}}\in\mathbf{R}^n$ is the state variable of node $i$ and $N$ is the number of the network nodes, $x_i(t-\tau(t))=(x_{i1}(t-\tau(t)),x_{i2}(t-\tau(t)),\cdots,x_{in}(t-\tau(t)))^{\mathrm{T}}$, $y_i(t-\tau(t))=(y_{i1}(t-\tau(t)),y_{i2}(t-\tau(t)),\cdots,y_{in}(t-\tau(t)))^{\mathrm{T}}$, $\tau(t)$ is the time-varying delays with $0\leqslant\tau(t)\leqslant\tau$. The matrix $\boldsymbol{H}$ is an arbitrary constant Hurwitz one (a matrix with negative real part eigenvalues). $g(\cdot):\mathbf{R}^n\to\mathbf{R}^n$ is a continuously differentiable function which determines the dynamical behavior of the nodes. $c>0$ is the coupling strength of the network. $\boldsymbol{\Gamma}\in\mathbf{R}^{n\times n}$ is a constant 0-1 matrix linking coupled variables. $\boldsymbol{A}=(a_{ij})_{N\times N}$ represent the coupling configurations of both networks.

Before stating our main results, we give some denotations, definitions and lemmas.

Let $\mathbf{R}^n$ denote the $n$-dimensional Euclidean space and $\mathbf{R}^{n\times m}$ be the space of $n\times m$ real matrices. $\boldsymbol{P}>0$ ($\boldsymbol{P}<0$) means matrix $\boldsymbol{P}$ is symmetrical and positive (negative) definite, $\boldsymbol{P}\geqslant0$ ($\boldsymbol{P}\leqslant0$) means matrix $\boldsymbol{P}$ is symmetrical and semi-positive (semi-negative) definite.

**Definition** 1 Network (1) and network (2) are said to achieve outer synchronization if

$$\lim_{t\to+\infty} \| y_i(t)-x_i(t) \| =0, i=1,2,\cdots,N \tag{3}$$

**Definition** 2[12]　$\boldsymbol{A}$ is said to be an reducible matrix, if there exist a permutation matrix $\boldsymbol{P}$, such that $\boldsymbol{A}=\boldsymbol{P}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{B} & \boldsymbol{C}\\ 0 & \boldsymbol{D}\end{bmatrix}\boldsymbol{P}$.

If there is no such a matrix $\boldsymbol{P}$, then, we said $\boldsymbol{A}$ is an irreducible matrix.

**Lemma** 1　Suppose that $\boldsymbol{A}=(\boldsymbol{A}_{ij})_{N\times N}$ is a real symmetric and irreducible matrix, where $\boldsymbol{A}_{ij}\geqslant 0 (i\neq j)$, $\boldsymbol{A}_{ii}=-\sum_{j=1,j\neq i}^{N}\boldsymbol{A}_{ij}$. Then, i) 0 is an eigenvalue of matrix $\boldsymbol{A}$ with multiplicity 1 and associated with eigenvector $(1,1,\cdots,1)^{\mathrm{T}}$; ii) all the other eigenvalues of $\boldsymbol{A}$ are real-valued and are strictly negative; iii) there exists a orthogonal matrix, $\boldsymbol{\Phi}=(\varphi_1,\varphi_2,\cdots,\varphi_N)$ such that $\boldsymbol{A}^{\mathrm{T}}\varphi_i=\lambda_i\varphi_i, i=1,2,\cdots,N$, where $\lambda_i$ are the eigenvalues of $\boldsymbol{A}$.

It is well known that the Linear Matrix Inequality (LMI) is an important tool for studying the behavior of dynamical systems. We also give the following lemmas on LMI.

**Lemma** 2[13]　The LMI $\begin{bmatrix}\boldsymbol{Q}(x) & \boldsymbol{S}(x)\\ \boldsymbol{S}^{\mathrm{T}}(x) & \boldsymbol{R}(x)\end{bmatrix}>0$, where $\boldsymbol{Q}(x)=\boldsymbol{Q}^{\mathrm{T}}(x)$, $\boldsymbol{R}(x)=\boldsymbol{R}^{\mathrm{T}}(x)$, and $\boldsymbol{S}(x)$ depends affinely on $x$, is equivalent to $\boldsymbol{R}(x)>0, \boldsymbol{Q}(x)-\boldsymbol{S}(x)\boldsymbol{R}^{-1}(x)\boldsymbol{S}^{\mathrm{T}}(x)>0$.

**Lemma** 3[14]　Assume that continuous functions $a: \Omega\to\mathbf{R}^{n_a}$, $b: \Omega\to\mathbf{R}^{n_b}$ and $\boldsymbol{U}\in\mathbf{R}^{n_a\times n_b}$, where an interval $\Omega\subset\mathbf{R}$. Then, for given matrices $\boldsymbol{X}\in\mathbf{R}^{n_a\times n_a}$; $\boldsymbol{Y}\in\mathbf{R}^{n_a\times n_b}$ and $\boldsymbol{Z}\in\mathbf{R}^{n_b\times n_b}$, the following inequality holds $2\int_{\Omega} a^{\mathrm{T}}(\mu)\boldsymbol{U}b(\mu)\,\mathrm{d}\mu \leqslant \int_{\Omega}\begin{bmatrix}a(\mu)\\ b(\mu)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}-\boldsymbol{U}\\ \boldsymbol{Y}^{\mathrm{T}}-\boldsymbol{U}^{\mathrm{T}} & \boldsymbol{Z}\end{bmatrix}\begin{bmatrix}a(\mu)\\ b(\mu)\end{bmatrix}\mathrm{d}\mu$, where $\begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}\\ \boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{Z}\end{bmatrix}\geqslant 0$.

## 2 Main results

To give some synchronization criteria, we always assume the follows.

(**H**1)　The network (1) is connected in the sense that there are no isolated clusters, that is, $\boldsymbol{A}=(a_{ij})_{n\times n}$ is an irreducible matrix, where $a_{ij}$ is defined as follows: if there is a connection between node $i$ and node $j (j\neq i)$, then $a_{ij}=a_{ji}=1$; otherwise, $a_{ij}=a_{ji}=0$ $(j\neq i)$, and the diagonal elements of matrix $\boldsymbol{A}$ are defined by $a_{ii}=-\sum_{j=1,j\neq i}^{N}a_{ij}, i=1,2,\cdots,N$.

Clearly, $\boldsymbol{A}$ is a real symmetric and irreducible matrix. By Lemma 1, we also suppose that

(**H**2)　$\lambda_i, i=1,2,\cdots,N$ are the eigenvalues of $\boldsymbol{A}$ and $0=\lambda_1>\lambda_2\geqslant\lambda_3\geqslant\cdots\geqslant\lambda_N$.

**Theorem** 1　Let (H1) and (H2) hold and $0\leqslant\dot{\tau}(t)\leqslant\sigma<1$. If there exist matrices $\boldsymbol{M}_i>0, \boldsymbol{S}_i>0$, such that

$$\begin{bmatrix}\boldsymbol{U}_i & \lambda_i c\boldsymbol{M}_i\boldsymbol{\Gamma}\\ \lambda_i c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i & -(1-\sigma)\boldsymbol{S}_i\end{bmatrix}<0, i=2,\cdots,N \tag{4}$$

Where $\boldsymbol{U}_i=\boldsymbol{M}_i\boldsymbol{H}+\boldsymbol{H}^{\mathrm{T}}\boldsymbol{M}_i+\boldsymbol{S}_i$, then the outer synchronization between the drive network (1) and the response one (2) is achieved.

**Proof**　Letting $e_i=y_i-x_i, i=1,2,\cdots N$, and linearizing the error system around $x_i$, we get

$$\dot{e}_i(t)=\boldsymbol{H}e_i(t)+c\textstyle\sum_{j=1}^{N}a_{ij}\boldsymbol{\Gamma}e_j(t-\tau(t)), i=1,2,\cdots,N \tag{5}$$

Equation (5) can be written as

$$\dot{\boldsymbol{e}}(t)=\boldsymbol{H}\boldsymbol{e}(t)+c\boldsymbol{\Gamma}\boldsymbol{e}(t-\tau(t))\boldsymbol{A}^{\mathrm{T}} \tag{6}$$

Let $\boldsymbol{e}(t)=(e_1(t),e_2(t),\cdots,e_N(t))\in\mathbf{R}^{n\times N}$ and $\boldsymbol{e}(t-\tau(t))=[e_1(t-\tau(t)),e_2(t-\tau(t)),\cdots,e_N(t-\tau(t))]$.

According to Lemma 1, there exists a orthogonal matrix $\boldsymbol{\Phi}=(\varphi_1,\varphi_2,\cdots,\varphi_N)\in\mathbf{R}^{N\times N}$ such that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Phi}=\boldsymbol{\Phi}\boldsymbol{\Lambda}$ with $\boldsymbol{\Lambda}=diag(\lambda_1,\lambda_2,\cdots,\lambda_N)$. Take a nonsingular transform

$$\boldsymbol{e}(t)\boldsymbol{\Phi}=\boldsymbol{\delta}(t)=(\delta_1(t),\delta_2(t),\cdots,\delta_N(t))\in\mathbf{R}^{n\times N} \tag{7}$$

From (7), we have the following matrix equation: $\dot{\boldsymbol{\delta}}(t)=\boldsymbol{H}\boldsymbol{\delta}(t)+c\boldsymbol{\Gamma}\boldsymbol{\delta}(t-\tau(t))\boldsymbol{\Lambda}$, where $\boldsymbol{\delta}(t-\tau(t))=[\delta_1(t-\tau(t)),\delta_2(t-\tau(t)),\cdots,\delta_N(t-\tau(t))]$, that is

$$\dot{\delta}_i(t)=\boldsymbol{H}\delta_i(t)+c\lambda_i\boldsymbol{\Gamma}\delta_i(t-\tau(t)), i=1,2,\cdots,N \tag{8}$$

Thus, the outer synchronization between the drive network (1) and the response one (2) is equivalent to stability of the zero solution of system (8). Our objective is to given the stability of the origin of the error network (8), i.e., $\lim_{t\to\infty}\|e_i(t)\|=0$. Note that $\lambda_1=0$ corresponds to the synchronization of the system states (3). Then the drive network (1) and the response network (2) with time-varying delays achieve the outer synchronization if the following $N-1$ pieces of $n$-dimensional linear time-varying delayed differential equations are asymptotically stable.

$$\dot{\delta}_i(t)=\boldsymbol{H}\delta_i(t)+c\lambda_i\boldsymbol{\Gamma}\delta_i(t-\tau(t)),i=2,3,\cdots,N \tag{9}$$

In the following, we shall prove that system (9) are asymptotically stable. Select Lyapunov functionals

$$V(\delta_i(t))=\delta_i^{\mathrm{T}}(t)\boldsymbol{M}_i\delta_i(t)+\int_{t-\tau(t)}^{t}\delta_i^{\mathrm{T}}(u)\boldsymbol{S}_i\delta_i(u)\mathrm{d}u \tag{10}$$

The derivative of $V(\delta_i(t))$ along the solution of the $i$ th($i=2,\cdots,N$) equation in system (9) is

$$\begin{aligned}\dot{V}(\delta_i(t))=&\dot{\delta}_i^{\mathrm{T}}(t)\boldsymbol{M}_i\delta_i(t)+\delta_i^{\mathrm{T}}(t)\boldsymbol{M}_i\dot{\delta}_i(t)+\delta_i^{\mathrm{T}}(t)\boldsymbol{S}_i\delta_i(t)-(1-\dot{\tau}(t))\delta_i^{\mathrm{T}}(t-\tau(t))\boldsymbol{S}_i\delta_i(t-\tau(t))=\\&\delta_i^{\mathrm{T}}(t)\boldsymbol{U}_i\delta_i(t)+2\delta_i^{\mathrm{T}}(t)\lambda_ic\boldsymbol{M}_i\boldsymbol{\Gamma}\delta_i(t-\tau(t))-(1-\dot{\tau}(t))\delta_i^{\mathrm{T}}(t-\tau(t))\boldsymbol{S}_i\delta_i(t-\tau(t))\leqslant\delta_i^{\mathrm{T}}(t)\boldsymbol{U}_i\delta_i(t)+\\&2\delta_i^{\mathrm{T}}(t)\lambda_ic\boldsymbol{M}_i\boldsymbol{\Gamma}\delta_i(t-\tau(t))-(1-\sigma)\delta_i^{\mathrm{T}}(t-\tau(t))\boldsymbol{S}_i\delta_i(t-\tau(t))=\\&\begin{bmatrix}\delta_i(t)\\\delta_i(t-\tau(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{U}_i&\lambda_ic\boldsymbol{M}_i\boldsymbol{\Gamma}\\\lambda_ic\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i&-(1-\sigma)\boldsymbol{S}_i\end{bmatrix}\begin{bmatrix}\delta_i(t)\\\delta_i(t-\tau(t))\end{bmatrix}\end{aligned}$$

Letting $f_i(t)=\begin{bmatrix}\delta_i(t)\\\dfrac{-\lambda_ic\boldsymbol{S}_i^{-1}\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i}{1-\sigma}\delta_i(t)+\delta_i(t-\tau(t))\end{bmatrix}\in\mathbf{R}^{2n}$, we have

$$\begin{aligned}&\begin{bmatrix}\delta_i(t)\\\delta_i(t-\tau(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{U}_i&\lambda_ic\boldsymbol{M}_i\boldsymbol{\Gamma}\\\lambda_ic\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i&-(1-\sigma)\boldsymbol{S}_i\end{bmatrix}\begin{bmatrix}\delta_i(t)\\\delta_i(t-\tau(t))\end{bmatrix}=f_i^{\mathrm{T}}(t)\begin{bmatrix}\boldsymbol{U}_i+\dfrac{\lambda_i^2c^2\boldsymbol{M}_i\boldsymbol{\Gamma}\boldsymbol{S}_i^{-1}\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i}{1-\sigma}&0\\0&-(1-\sigma)\boldsymbol{S}_i\end{bmatrix}f_i(t)\leqslant\\&\delta_i^{\mathrm{T}}(t)\left(\boldsymbol{U}_i+\frac{\lambda_i^2c^2\boldsymbol{M}_i\boldsymbol{\Gamma}\boldsymbol{S}_i^{-1}\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i}{1-\sigma}\right)\delta_i(t)\end{aligned}$$

From the Schur complements (Lemma 2), the LMI (4) is equivalent to $\boldsymbol{U}_i+\dfrac{\lambda_i^2c^2\boldsymbol{M}_i\boldsymbol{\Gamma}\boldsymbol{S}_i^{-1}\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i}{1-\sigma}<0$.

Thus, we have $\dot{V}(\delta_i(t))$ is negative definite. According to Lyapunov stability theory, we know that systems (9) is asymptotically stable. So, we have the outer synchronization between the drive network (1) and the response one (2) with time-varying delays. The proof is completed.

Letting $\boldsymbol{M}_i=\boldsymbol{M},\boldsymbol{S}_i=\boldsymbol{S},i=2,3,\cdots,N$ in Theorem 1, we have

**Corollary** 1 Let (H1) and (H2) hold and $0\leqslant\dot{\tau}(t)\leqslant\sigma<1$. If there exist two positive-definite matrices $\boldsymbol{M}$, $\boldsymbol{S}>0$ such that

$$\begin{bmatrix}\boldsymbol{MH}+\boldsymbol{H}^{\mathrm{T}}\boldsymbol{M}+\boldsymbol{S}&\lambda_Nc\boldsymbol{M\Gamma}\\\lambda_Nc\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}&-(1-\sigma)\boldsymbol{S}\end{bmatrix}<0 \tag{11}$$

then the outer synchronization between the drive network (1) and the response one (2) is achieved.

In Theorem 1, replacing the constant matrices $\boldsymbol{M}_i$ by $\boldsymbol{M}(t)$ and $\boldsymbol{S}_i$ by $\boldsymbol{S}$ and noting that there is an additional term $\delta_i{}^{\mathrm{T}}(t)\dot{\boldsymbol{M}}(t)\delta_i(t)$ in $\dot{V}(\delta_i(t))$, we easily obtain the follows.

**Corollary** 2 Let (H1) and (H2) hold and $0\leqslant\dot{\tau}(t)\leqslant\sigma<1$. If there exist a constant $\varepsilon>0$ and a matrix $\boldsymbol{S}>0$, such that the following Riccati equation

$$\dot{\boldsymbol{M}}(t)+\boldsymbol{U}(t)+\frac{\lambda_N^2c^2\boldsymbol{M}(t)\boldsymbol{\Gamma}\boldsymbol{S}^{-1}\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}(t)}{1-\sigma}+\varepsilon\boldsymbol{I}=0 \tag{12}$$

has a positive-definite and symmetric solution $\boldsymbol{M}(t)>0$, $t\in[t_0,\infty)$, then the outer synchronization between the drive network (1) and the response one (2) is achieved.

It is noted that the derivative of the time-varying delays isn't always non-negative in the real world. In the following, we consider the derivative of the time-varying delays is non-positive situation.

**Theorem** 2 Let (H1) and (H2) hold. Assume that $\dot{\tau}(t)\leqslant0$ and $\tau(t)\leqslant\tau$ for some $0<\tau<\infty$. If there exist common matrices $\boldsymbol{M}_k>0,\boldsymbol{S}_k>0$, $\boldsymbol{X}_k$, $\boldsymbol{Y}_k$ and $\boldsymbol{Z}_k$ such that

$$\begin{bmatrix} \boldsymbol{U}_k & \lambda_k c\boldsymbol{M}_k\boldsymbol{\Gamma}-\boldsymbol{Y}_k & \tau\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Z}_k \\ \lambda_k c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_k-\boldsymbol{Y}_k^{\mathrm{T}} & -\boldsymbol{S}_k & \tau\lambda_k c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Z}_k \\ \tau\boldsymbol{Z}_k\boldsymbol{H} & \tau\lambda_k c\boldsymbol{Z}_k\boldsymbol{\Gamma} & -\tau\boldsymbol{Z}_k \end{bmatrix}<0 \tag{13}$$

where

$$\begin{bmatrix} \boldsymbol{X}_k & \boldsymbol{Y}_k \\ \boldsymbol{Y}_k^{\mathrm{T}} & \boldsymbol{Z}_k \end{bmatrix}\geqslant 0 \tag{14}$$

and $\boldsymbol{U}_k=\boldsymbol{M}_k\boldsymbol{H}+\boldsymbol{H}^{\mathrm{T}}\boldsymbol{M}_k+\tau\boldsymbol{X}_k+\boldsymbol{Y}_k+\boldsymbol{Y}_k^{\mathrm{T}}+\boldsymbol{S}_k$ for $k=2,3,\cdots,N$, then the outer synchronization between the drive network (1) and the response one (2) is achieved.

**Proof**　For the $k$ th subsystem of (9), define the following Lyapunov functional

$$V_k(\delta_k(t))=V_{k1}+V_{k2}+V_{k3} \tag{15}$$

where $V_{k1}=\delta_k^{\mathrm{T}}(t)\boldsymbol{M}_k\delta_k(t)$, $V_{k2}=\int_{-\tau(t)}^{0}\int_{t+\varphi}^{t}\dot{\delta}_k^{\mathrm{T}}(\mu)\boldsymbol{Z}_k\dot{\delta}_k(\mu)\mathrm{d}\mu\mathrm{d}\varphi$, $V_{k3}=\int_{t-\tau(t)}^{t}\delta_k^{\mathrm{T}}(\mu)\boldsymbol{S}_k\delta_k(\mu)\mathrm{d}\mu$.

The $k$ th ($k=2,\cdots,N$) equation in system (9) can be written as

$$\dot{\delta}_k(t)=(\boldsymbol{H}+\lambda_k c\boldsymbol{\Gamma})\delta_k(t)-\lambda_k c\boldsymbol{\Gamma}\int_{t-\tau(t)}^{t}\dot{\delta}_k(\mu)\mathrm{d}\mu \tag{16}$$

Thus, the derivative of $V_{k1}$ satisfies

$$\dot{V}_{k1}=\delta_k^{\mathrm{T}}(t)[\boldsymbol{M}_k(\boldsymbol{H}+\lambda_k c\boldsymbol{\Gamma})+(\boldsymbol{H}+\lambda_k c\boldsymbol{\Gamma})^{\mathrm{T}}\boldsymbol{M}_k]\delta_k(t)-2\delta_k^{\mathrm{T}}(t)c\lambda_k\boldsymbol{M}_k\boldsymbol{\Gamma}\int_{t-\tau(t)}^{t}\dot{\delta}_k(\mu)\mathrm{d}\mu$$

Define $a(\cdot)$, $b(\cdot)$, and $\boldsymbol{U}$ in Lemma 3 as $a(\mu):=\delta_k(t)$, $b(\mu):=\dot{\delta}_k(\mu)$, and $\boldsymbol{U}=\boldsymbol{M}_k c\lambda_k\boldsymbol{\Gamma}$ for $\tau(t)\leqslant\tau$. Combining Lemma 3 and LMI (14), we get

$$\dot{V}_{k1}\leqslant\delta_k^{\mathrm{T}}(t)[\boldsymbol{M}_k(\boldsymbol{H}+\lambda_k c\boldsymbol{\Gamma})+(\boldsymbol{H}+\lambda_k c\boldsymbol{\Gamma})^{\mathrm{T}}\boldsymbol{M}_k+\tau\boldsymbol{X}_k]\delta_k(t)+2\delta_k^{\mathrm{T}}(t)(\boldsymbol{Y}_k-\lambda_k c\boldsymbol{M}_k\boldsymbol{\Gamma})\int_{t-\tau(t)}^{t}\dot{\delta}_k(\mu)\mathrm{d}\mu+$$
$$\int_{t-\tau(t)}^{t}\dot{\delta}_k^{\mathrm{T}}(\mu)\boldsymbol{Z}_k\dot{\delta}_k(\mu)\mathrm{d}\mu\leqslant\delta_k^{\mathrm{T}}(t)(\boldsymbol{M}_k\boldsymbol{H}+\boldsymbol{H}^{\mathrm{T}}\boldsymbol{M}_k+\tau\boldsymbol{X}_k+\boldsymbol{Y}_k+\boldsymbol{Y}_k^{\mathrm{T}})\delta_k(t)+2\delta_k^{\mathrm{T}}(t)(\lambda_k c\boldsymbol{M}_k\boldsymbol{\Gamma}-\boldsymbol{Y}_k)\delta_k(t-\tau(t))+$$
$$\int_{t-\tau(t)}^{t}\dot{\delta}_k^{\mathrm{T}}(\mu)\boldsymbol{Z}_k\dot{\delta}_k(\mu)\mathrm{d}\mu$$

Moreover　$\dot{V}_{k2}=\dot{\tau}(t)\int_{t-\tau(t)}^{t}\dot{\delta}_k^{\mathrm{T}}(\mu)\boldsymbol{Z}_k\dot{\delta}_k(\mu)\mathrm{d}\mu+\tau(t)\dot{\delta}_k^{\mathrm{T}}(\mu)\boldsymbol{Z}_k\dot{\delta}_k(\mu)-\int_{t-\tau(t)}^{t}\dot{\delta}_k^{\mathrm{T}}(\mu)\boldsymbol{Z}_k\dot{\delta}_k(\mu)\mathrm{d}\mu\leqslant$

$$(\dot{\tau}(t)-1)\int_{t-\tau(t)}^{t}\dot{\delta}_k^{\mathrm{T}}(\mu)\boldsymbol{Z}_k\dot{\delta}_k(\mu)\mathrm{d}\mu+\tau[\boldsymbol{H}\delta_k(t)+\lambda_k c\boldsymbol{\Gamma}\delta_k(t-\tau(t))]^{\mathrm{T}}\boldsymbol{Z}_k[\boldsymbol{H}\delta_k(t)+\lambda_k c\boldsymbol{\Gamma}\delta_k(t-\tau(t))]$$

$$\dot{V}_{k3}=\delta_k^{\mathrm{T}}(t)\boldsymbol{S}_k\delta_k(t)-(1-\dot{\tau}(t))\delta_k^{\mathrm{T}}(t-\tau(t))\boldsymbol{S}_k\delta_k(t-\tau(t))$$

Letting $\zeta(t)=\begin{bmatrix}\delta_k(t)\\ \delta_k(t-\tau(t))\end{bmatrix}\in\mathbf{R}^{2n}$, we have the derivative of $V_k$ is $\dot{V}_k=\dot{V}_{k1}+\dot{V}_{k2}+\dot{V}_{k3}\leqslant\zeta^{\mathrm{T}}(t)\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{C}\end{bmatrix}\zeta(t)$, where $\boldsymbol{A}:=\boldsymbol{U}_k+\tau\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Z}_k\boldsymbol{H}$, $\boldsymbol{B}:=\lambda_k c\boldsymbol{M}_k\boldsymbol{\Gamma}-\boldsymbol{Y}_k+\tau\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Z}_k\lambda_k c\boldsymbol{\Gamma}$, $\boldsymbol{C}:=-\boldsymbol{S}_k+\tau\lambda_k^2c^2\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Z}_k\boldsymbol{\Gamma}$.

It follows from Lemma 2 that (13) is equivalent to the following inequality: $\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{C}\end{bmatrix}<0$. Therefore, $\dot{V}_k$ is negative definite. Using Lyapunov stability theory, we conclude that system (9) is asymptotically stable. This completes the proof.

In Theorem 2, we choose Lyapunov functionals as follows

$$V_k(\delta_k(t))=V_{k1}+V_{k2}+V_{k3} \tag{17}$$

where $V_{k1}=\delta_k^{\mathrm{T}}(t)\boldsymbol{M}\delta_k(t)$, $V_{k2}=\int_{-\tau(t)}^{0}\int_{t+\varphi}^{t}\dot{\delta}_k^{\mathrm{T}}(\mu)\boldsymbol{Z}\dot{\delta}_k(\mu)\mathrm{d}\mu\mathrm{d}\varphi$, $V_{k3}=\int_{t-\tau(t)}^{t}\delta_k^{\mathrm{T}}(\mu)\boldsymbol{S}\delta_k(\mu)\mathrm{d}\mu$, where $k=2,\cdots,N$. Then, we easily obtain

**Corollary** 3　Suppose that the time-varying delay $\dot{\tau}(t)\leqslant 0$ and $\tau(t)\leqslant\tau$ for some $0<\tau<\infty$. If there exist common matrices $\boldsymbol{M}>0$, $\boldsymbol{S}>0$, $\boldsymbol{M}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ such that

$$\begin{bmatrix} \boldsymbol{Q} & \lambda_k c\boldsymbol{M}\boldsymbol{\Gamma}-\boldsymbol{Y} & \tau\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Z} \\ \lambda_i c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}-\boldsymbol{Y}^{\mathrm{T}} & -\boldsymbol{S} & \tau\lambda_i c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Z} \\ \tau\boldsymbol{Z}\boldsymbol{H} & \tau\lambda_i c\boldsymbol{Z}\boldsymbol{\Gamma} & -\tau\boldsymbol{Z} \end{bmatrix}<0 \tag{18}$$

where

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \\ \boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{Z} \end{bmatrix} \geqslant 0 \tag{19}$$

and $\boldsymbol{Q}=\boldsymbol{MH}+\boldsymbol{H}^{\mathrm{T}}\boldsymbol{M}+\tau\boldsymbol{X}+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}+\boldsymbol{S}$ for $k=2,3,\cdots,N$, then the outer synchronization between the drive network (1) and the response one (2) is achieved.

## 3 Example

In this section, a numerical example is used to show the effectiveness of the proposed synchronization criteria derived in the Section 3.

**Example** 1 Consider a 3-nodes the driving network (1) and the responding network (2). The Hurwitz matrix is $\boldsymbol{H}=\mathrm{diag}(-6,-7,-8)$, and the eigenvalues of $\boldsymbol{A}$ are $\lambda_1=0, \lambda_2=-1, \lambda_3=-3$. The coupling configurations matrix of both networks with three nodes in this case is given by $\boldsymbol{A}=\begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. We assume that the matrix $\boldsymbol{\Gamma}$ is given by $\boldsymbol{\Gamma}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

In the following, we analyze the outer synchronization between the drive network (1) and the response one (2) for two case in different coupling strength parameter $c$ and time-varying delay function $\tau(t)$.

**Case** 1: $c=0.2$ and $\tau(t)=3-\frac{1}{4}\mathrm{e}^{-t}$.

In Theorem 1, conditions (H1), (H2) are satisfied and $0\leqslant\dot{\tau}(t)=\frac{1}{4}\mathrm{e}^{-t}\leqslant\frac{1}{4}=\sigma$ for $t\geqslant 0$. Furthermore, we can check the LMI (4) with $\boldsymbol{M}_2=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $\boldsymbol{M}_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\boldsymbol{S}_2=\begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$, $\boldsymbol{S}_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{U}_2=\begin{bmatrix} -4.6 & 0 & 0 \\ 0 & -15.6 & 0 \\ 0 & 0 & -37.6 \end{bmatrix}$, $\boldsymbol{U}_3=\begin{bmatrix} -2 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -19 \end{bmatrix}$.

From Theorem 1, we see that the drive network (1) and the response network (2) achieve the outer synchronization.

**Case** 2: $c=0.3$ and $\tau(t)=\frac{6}{t+8}$.

In Theorem 2, conditions (H1), (H2) aresatisfied and $\dot{\tau}(t)=-\frac{6}{(t+8)^2}\leqslant 0$, $\tau(t)\leqslant 0.75=\tau$ for $t\geqslant 0$. Furthermore, we can check the LMI (13) with $\boldsymbol{M}_2=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $\boldsymbol{M}_3=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 13 \end{bmatrix}$, $\boldsymbol{S}_2=\begin{bmatrix} 1.327 & 0 & 0 \\ 0 & 1.327 & 0 \\ 0 & 0 & 1.327 \end{bmatrix}$, $\boldsymbol{S}_3=\begin{bmatrix} 1.509 & 0 & 0 \\ 0 & 1.509 & 0 \\ 0 & 0 & 1.509 \end{bmatrix}$, $\boldsymbol{X}_2=\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$, $\boldsymbol{X}_3=\begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $\boldsymbol{Y}_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{Y}_3=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{Z}_2=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\boldsymbol{Z}_3=\begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$, $\boldsymbol{U}_2=\begin{bmatrix} -3.487 & 0 & 0 \\ 0 & -16.596 & 0 \\ 0 & 0 & -29.362 \end{bmatrix}$, $\boldsymbol{U}_3=\begin{bmatrix} -5.392 & 0 & 0 \\ 0 & -25.964 & 0 \\ 0 & 0 & -62.241 \end{bmatrix}$.

From Theorem 2, we see that the drive network (1) and the response network (2) achieve the outer synchronization.

In the above example, since time delays are time-varying and we take different matrix parameters, our criteria arefiexible and easily verified by LMI Toolbox in Matlab.

## 4 Conclusion

In this paper, the outer synchronization between two coupled complex networks with time-varying delays were considered. We investigated the case that the topology structure are frequently varied with time. Several theorems with regard to judging the outer synchronization between two complex networks have been obtained. At last, a numerical example is given to illustrate the theoretical results.

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# 具有时变时滞的两个复杂动态网络间的外同步


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**摘要:**本文讨论了一类具有时变时滞的驱动-响应网络的外同步问题。以线性矩阵不等式(LMI)和 Lyapunov 泛函方法,获得了该两个复杂动态网络间达到外同步的判据。即当系统参数满足下列条件之一:即当(1)$0\leqslant\dot{\tau}(t)\leqslant\sigma<1$, $\boldsymbol{M}_i>0,\boldsymbol{S}_i>0$, $\begin{bmatrix}\boldsymbol{U}_i(t) & \lambda_i c M_i \boldsymbol{\Gamma}\\ \lambda_i c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_i & -(1-\sigma)\boldsymbol{S}_i\end{bmatrix}<0$, $i=2,\cdots,N$;(2)$\dot{\tau}(t)\leqslant0$, $\tau(t)\leqslant\tau$, $0<\tau<\infty$, $\boldsymbol{M}_k>0,S_k>0$, $\begin{bmatrix}\boldsymbol{U}_k & \lambda_k c\boldsymbol{M}_k\boldsymbol{\Gamma}-\boldsymbol{Y}_k & \tau\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Z}_k\\ \lambda_k c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{M}_k-\boldsymbol{Y}_k^{\mathrm{T}} & -\boldsymbol{S}_k & \tau\lambda_k c\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Z}_k\\ \tau\boldsymbol{Z}_k\boldsymbol{H} & \tau\lambda_k c\boldsymbol{Z}_k\boldsymbol{\Gamma} & -\tau\boldsymbol{Z}_k\end{bmatrix}<0$, $k=2,\cdots,N$,则驱动-响应网络达到外同步。最后用数值例子验证了结论的有效性。



**关键词:**复杂网络;外同步;时变时滞;Lyapunov 泛函

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