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Pachpatte 离散不等式的一个推广*

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摘要:本文主要研究了 Pachpatte 不等式的推广及其类似不等式,也就是经典的 Hilbert 不等式的变式。通过引进一λ 齐次函数 $K(x,y)$和两对共轭指数(p,q),(r,s),$(1/p)+(1/p)=1$,$(1/r)+(1/s)=1$,经过巧妙配方,再运用一些经典的不等式(例如 Hölder 不等式、Young 不等式与 Jensen 不等式)技巧和一定的实分析方法来估算权函数,建立了一系列 Pachpatte 离散不等式的推广及类似形式,包括非负凸、次可乘的可测实值函数下的各种不等式.该结论综合运用了 Hilbert 不等式和 Pachpatte 不等式的推演技巧,将以前不含共轭指数或只含一对共轭指数的 Pachpatte 不等式推广到含两对共轭指数与参量化的不等式,统一了部分已有文献的研究成果,使 Pachpatte 不等式的研究上升到一个更高的层次。作为应用,对齐一λ 次函数 $K(x,y)$取了 2 个特殊的函数得到了一些有趣的不等式。

关键词:Hölder 不等式;Young 不等式;Jensen 不等式

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1 研究背景

1934 年,D. Hilbert 证明了如下经典的不等式[1] $\sum\limits_n\sum\limits_m \frac{a_m b_n}{m+n}\leqslant \pi\left(\sum\limits_n a_n^2\right)^{1/2}\left(\sum\limits_n b_n^2\right)^{1/2}$,这就是著名的 Hilbert 不等式,此后大量工作如雨后春笋涌现。1998 年,B. G. Pachpatte 得到若干 Hilbert 不等式的相似不等式[2],设 $p\geqslant 1$, $q\geqslant 1$,$\{a_m\}$,$\{b_n\}$为非负实数列,$A_m=\sum_{s=1}^m a_s$ 和 $B_n=\sum_{t=1}^n b_t$ 。则

$$\sum_{m=1}^k\sum_{n=1}^r \frac{A_m^p B_n^q}{m+n}\leqslant C(p,q,k,r)\left\{\left(\sum_{m=1}^k (k-m+1)\left(A_m^{p-1}a_m\right)^2\right)\right\}^{1/2}\times\left\{\sum_{n=1}^r (r-n+1)\left(B_n^{q-1}b_n\right)^2\right\}^{1/2}$$

除非$\{a_m\}$或$\{b_n\}$恒为 0,其中 $C(p,q,k,r)=\frac{1}{2}pq\sqrt{kr}$。

设$\{a_m\}$,$\{b_n\}$,A_m,B_n 如上所述,$\{p_m\}$和$\{q_n\}$为正值数列,$P_m=\sum_{s=1}^m p_s$,$Q_n=\sum_{t=1}^n q_t$,φ 和 ψ 为定义在 $\mathbf{R}_+=[0,\infty)$上的非负凸、次可乘的可测实值函数,则

$$\sum_{m=1}^k\sum_{n=1}^r \frac{\varphi(A_m)\psi(B_n)}{m+n}\leqslant M(k,r)\left\{\sum_{m=1}^k (k-m+1)\left[p_m\varphi\left(\frac{a_m}{p_m}\right)\right]^2\right)^{1/2}\times\left\{\sum_{n=1}^r (r-n+1)\left[q_n\psi\left(\frac{b_n}{q_n}\right)\right]^2\right\}^{1/2}$$

其中 $M(k,r)=\frac{1}{2}\left(\sum\limits_{m=1}^k\left[\frac{\varphi(P_m)}{P_m}\right]^2\right)^{1/2}\left(\sum\limits_{n=1}^r\left[\frac{\psi(Q_n)}{Q_n}\right]^2\right)^{1/2}$。这 2 个不等式称为 Pachpatte 不等式。随后,很多学者对其进行广泛研究,其众多相关不等式见诸于各类文献中[3-8]。

本文主要目的是应用分析方法和不等式理论来建立更一般的 Pachpatte 离散不等式和相关不等式。作为应用,考虑了某些特定情况下的不等式。

2 主要结果

若实可测函数 $K(x,y)$满足 $K(ux,uy)=u^{-\lambda}K(x,y)$,其中 $\lambda>0$,$u>0$,$(x,y)\in(0,\infty)\times(0,\infty)$,则称 $K(x,y)$为一λ 齐次函数。

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引理 1　若 $r>1,\frac{1}{r}+\frac{1}{w}=1,\lambda>0,K(x,y)(\geqslant 0)$ 为 $-\lambda$ 齐次函数，权函数 $\overline{w}(\lambda,w,x),\overline{w}(\lambda,r,y)$ 定义为 $\overline{w}_1(\lambda,w,x):=\int_0^{\infty}K(x,y)\frac{x^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{w}}}\mathrm{d}y,x\in(0,\infty),\overline{w}_2(\lambda,r,y):=\int_0^{\infty}K(x,y)\frac{x^{\frac{\lambda}{w}}}{y^{1-\frac{\lambda}{r}}}\mathrm{d}x,y\in(0,\infty),C(\lambda,r):=\int_0^{\infty}K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u$。

则有

$$\overline{w}_1(\lambda,w,x)=\overline{w}_2(\lambda,r,y)=C(\lambda,r)\tag{1}$$

证明　作变换 $u=\frac{x}{y}$，则 $\overline{w}_1(\lambda,w,x)=\int_0^{\infty}K\left(\frac{x}{y}\cdot y,y\right)\frac{\left(\frac{x}{y}\cdot y\right)^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{w}}}\cdot\frac{-y^2}{x}\mathrm{d}\frac{x}{y}=\int_0^{\infty}K(u,1)x^{\frac{\lambda}{r}-1}\mathrm{d}u=C(\lambda,r)$，类似地，令 $u=\frac{x}{y}$ 得 $\overline{w}_2(\lambda,r,y)=\int_0^{\infty}K\left(\frac{x}{y}\cdot y,y\right)\frac{y^{\frac{\lambda}{w}}}{\left(\frac{x}{y}\cdot y\right)^{1-\frac{\lambda}{r}}}\cdot y\mathrm{d}\frac{x}{y}=\int_0^{\infty}K(u,1)x^{\frac{\lambda}{r}-1}\mathrm{d}u=C(\lambda,r)$。因此(1)式正确。　　证毕

引理 2　设 $r>1,\frac{1}{r}+\frac{1}{w}=1,\lambda>0,K(x,y)(\geqslant 0)$ 为 $-\lambda$ 齐次函数，$K(x,y)\frac{1}{x^{1-\frac{\lambda}{r}}}$ 关于 x 在 $(0,\infty)$ 上递减，$K(x,y)\frac{1}{y^{1-\frac{\lambda}{w}}}$ 关于 y 在 $(0,\infty)$ 上递减，定义 $v_1(\lambda,w,m)$ 与 $v_2(\lambda,r,n)$ 为

$$v_1(\lambda,w,m):=\sum_{n=1}^{\infty}K(m,n)\frac{m^{\frac{\lambda}{r}}}{n^{1-\frac{\lambda}{w}}},m\in\mathbf{N}_+,v_2(\lambda,r,n):=\sum_{m=1}^{\infty}K(m,n)\frac{m^{\frac{\lambda}{w}}}{n^{1-\frac{\lambda}{r}}},n\in\mathbf{N}_+$$

则有

$$v_1(\lambda,\omega,m)<C(\lambda,r),v_2(\lambda,r,n)<C(\lambda,r)\tag{2}$$

证明　由函数单调性和齐次性，注意到(1)式有 $v_1(\lambda,w,m)=\sum_{n=1}^{\infty}K(m,n)\frac{m^{\frac{\lambda}{r}}}{n^{1-\frac{\lambda}{w}}}<\int_0^{\infty}K(m,y)\frac{m^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{w}}}\mathrm{d}y$。令 $t=\frac{y}{m}$，则 $v_1(\lambda,w,m)<\int_0^{\infty}K(t,1)t^{\frac{\lambda}{r}-1}\mathrm{d}t=C(\lambda,r)$。类似地，$v_2(\lambda,r,n)<C(\lambda,r)$，于是(2)式正确。　　证毕

定理 3　设 $a_s\geqslant 0,b_t\geqslant 0,A_m=\sum_{s=1}^{m}a_s,B_n=\sum_{t=1}^{n}b_t,\alpha,\beta\geqslant 1,p,r>1,\frac{1}{p}+\frac{1}{q}=1,\frac{1}{r}+\frac{1}{w}=1$。若 $K(x,y)\geqslant 0$ 为 $-\lambda$ 齐次可测函数，权函数 $C(\lambda,r)=\int_0^{\infty}K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u$ 为依赖于 λ,r 的正数，$K_{\lambda}(x,y)\frac{1}{x^{1-\lambda/r}}$ 关于 x 在 $(0,\infty)$ 上递减，$K_{\lambda}(x,y)\frac{1}{y^{1-\lambda/w}}$ 关于 y 在 $(0,\infty)$ 上递减，则

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(m,n)A_m^{\alpha}B_n^{\beta}}{m+n}\leqslant\frac{\alpha\beta}{2}C(\lambda,r)\left\{\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{r}\right)-1}\widetilde{A}_m^{p}\right\}^{1/p}\times\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\widetilde{B}_n^{q}\right\}^{1/q}$$

除非 $\{a_m\}$ 或 $\{b_n\}$ 恒为 0。其中 $\widetilde{A}_m=\left\{\sum_{s=1}^{m}(a_sA_s^{\alpha-1})^2\right\}^{1/2}$ 和 $\widetilde{B}_n=\left\{\sum_{t=1}^{n}(b_tB_t^{\beta-1})^2\right\}^{1/2}$。

证明　由不等式[9] $\left(\sum_{m=1}^{n}z_m\right)^{\alpha}\leqslant\alpha\sum_{m=1}^{n}z_m\left(\sum_{k=1}^{m}z_k\right)^{\alpha-1}$，其中 $a\geqslant 1$ 为常数，$z_m\geqslant 0(m=1,2,\cdots)$，易知 $A_m^{\alpha}\leqslant\alpha\sum_{s=1}^{m}a_sA_s^{\alpha-1},m=1,2,\cdots,B_n^{\beta}\leqslant\beta\sum_{t=1}^{n}b_tB_t^{\beta-1},n=1,2,\cdots$，对上 2 式应用 Hölder 不等式和 Young 不等式：$ab\leqslant\frac{a^P}{p}+\frac{b^q}{q}$，其中 $a\geqslant 0,b\geqslant 0,p>1,\frac{1}{p}+\frac{1}{q}=1$，得

$$A_m^{\alpha}B_n^{\beta}\leqslant\alpha\beta\left(\sum_{s=1}^{m}a_sA_s^{\alpha-1}\right)\left(\sum_{t=1}^{n}b_tB_t^{\beta-1}\right)\leqslant\alpha\beta\sqrt{mn}\left\{\sum_{s=1}^{m}(a_sA_s^{\alpha-1})^2\right\}^{1/2}\left\{\sum_{t=1}^{n}(b_tB_t^{\beta-1})^2\right\}^{1/2}\leqslant$$
$$\frac{1}{2}\alpha\beta(m+n)\left\{\sum_{s=1}^{m}(a_sA_s^{\alpha-1})^2\right\}^{1/2}\left\{\sum_{t=1}^{n}(b_tB_t^{\beta-1})^2\right\}^{1/2}$$

记 $\widetilde{A}_m=\left\{\sum_{s=1}^{m}(a_sA_s^{\alpha-1})^2\right\}^{1/2},\widetilde{B}_n=\left\{\sum_{t=1}^{n}(b_tB_t^{\beta-1})^2\right\}^{1/2}$。

由 Hölder 不等式，注意到引理 2，有

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(m,n)A_m^{\alpha}B_n^{\beta}}{m+n}\leqslant\frac{\alpha\beta}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\widetilde{A}_m\widetilde{B}_n=\frac{\alpha\beta}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\left[\widetilde{A}_m\frac{m^{\left(1-\frac{\lambda}{r}\right)/q}}{n^{\left(1-\frac{\lambda}{\omega}\right)/p}}\right]\left[\widetilde{B}_n\frac{n^{\left(1-\frac{\lambda}{\omega}\right)/p}}{m^{\left(1-\frac{\lambda}{r}\right)/q}}\right]\leqslant$$

$$\frac{\alpha\beta}{2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\widetilde{A}_m^p\frac{m^{(p-1)\left(1-\frac{\lambda}{r}\right)}}{n^{\left(1-\frac{\lambda}{\omega}\right)}}\right\}^{1/p}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\widetilde{B}_n^q\frac{n^{(q-1)\left(1-\frac{\lambda}{\omega}\right)}}{m^{\left(1-\frac{\lambda}{r}\right)}}\right\}^{1/q}=$$

$$\frac{\alpha\beta}{2}\left\{\sum_{m=1}^{\infty}v_1(\lambda,w,m)m^{p\left(1-\frac{\lambda}{r}\right)-1}\widetilde{A}_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}v_2(\lambda,r,n)n^{q\left(1-\frac{\lambda}{w}\right)-1}\widetilde{B}_n^q\right\}^{1/q}<$$

$$\frac{\alpha\beta}{2}C(\lambda,r)\left\{\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{r}\right)-1}\widetilde{A}_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\widetilde{B}_n^q\right\}^{1/q}$$

证毕

定理4　若 $p_i,q_i,\{a_m\},\{b_n\},A_m,B_n$ 和 $K(x,y)$ 如上定理所述，$\{p_m\}$ 和 $\{q_n\}$ 为正值数列，$P_m=\sum_{s=1}^{m}p_s,Q_n=\sum_{t=1}^{n}q_t$，$\varphi$ 和 ψ 为定义在 $\mathbf{R}_+=[0,\infty)$ 上的非负凸、次可乘的可测实值函数，则

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(m,n)\varphi(A_m)\psi(B_n)}{m+n}\leqslant\frac{1}{2}C(\lambda,r)\left\{\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{r}\right)-1}\varphi^2(m)\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\psi^2(n)\right\}^{1/q}$$

其中 $\varphi_1(m)=\dfrac{\varphi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\varphi\left(\dfrac{a_s}{p_s}\right)\right]^2\right\}^{1/2}$，$\psi_1(n)=\dfrac{\psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{m}\left[q_t\psi\left(\dfrac{b_t}{q_t}\right)\right]^2\right\}^{1/2}$。

证明　由定理4的假设，应用Jensen不等式和Hölder不等式，易知

$$\varphi(A_m)=\varphi\left(\frac{P_m\sum_{s=1}^{m}p_sa_s/p_s}{\sum_{s=1}^{m}p_s}\right)\leqslant\varphi(P_m)\varphi\left(\frac{\sum_{s=1}^{m}p_sa_s/p_s}{\sum_{s=1}^{m}p_s}\right)\leqslant$$

$$\frac{\varphi(P_m)}{P_m}\sum_{s=1}^{m}\left[p_s\varphi\left(\frac{a_s}{p_s}\right)\right]\leqslant\frac{\varphi(P_m)}{P_m}\sqrt{m}\left(\sum_{s=1}^{m}\left[p_s\varphi\left(\frac{a_s}{p_s}\right)\right]^2\right)^{1/2}$$

类似地，$\psi(B_n)\leqslant\dfrac{\varphi(Q_n)}{Q_n}\sqrt{n}\left\{\sum_{t=1}^{n}\left[q_t\psi\left(\dfrac{b_t}{q_t}\right)\right]^2\right\}^{1/2}$，应用Young不等式，有

$$\varphi(A_m)\psi(B_n)\leqslant\frac{m+n}{2}\frac{\varphi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\varphi\left(\frac{a_s}{p_s}\right)\right]^2\right\}^{1/2}\frac{\psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{n}\left[q_t\psi\left(\frac{b_t}{q_t}\right)\right]^2\right\}^{1/2}\tag{3}$$

记 $\varphi_1(m)=\dfrac{\varphi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\varphi\left(\dfrac{a_s}{p_s}\right)\right]^2\right\}^{1/2}$，$\psi_1(n)=\dfrac{\psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{n}\left[q_t\psi\left(\dfrac{b_t}{q_t}\right)\right]^2\right\}^{1/2}$。

(3)式两边乘以 $K(m,n)/(m+n)$，关于 m,n 求和，应用Hölder不等式和引理2，有

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(m,n)\varphi(A_m)\psi(B_n)}{m+n}\leqslant\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\varphi_1(m)\psi_1(n)=$$

$$\frac{\alpha\beta}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\left[\varphi_1(m)\frac{m^{\left(1-\frac{\lambda}{r}\right)/q}}{n^{\left(1-\frac{\lambda}{w}\right)/p}}\right]\left[\psi_1(n)\frac{n^{\left(1-\frac{\lambda}{w}\right)/p}}{m^{\left(1-\frac{\lambda}{r}\right)/q}}\right]\leqslant$$

$$\frac{1}{2}\left\{\sum_{m=1}^{\infty}v_1(\lambda,w,m)m^{p\left(1-\frac{\lambda}{r}\right)-1}\varphi^2(m)\right\}^{1/p}\left\{\sum_{n=1}^{\infty}v_2(\lambda,r,n)n^{q\left(1-\frac{\lambda}{w}\right)-1}\psi^2(n)\right\}^{1/q}<$$

$$\frac{1}{2}C(\lambda,r)\left\{\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{r}\right)-1}\varphi^2(m)\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\psi^2(n)\right\}^{1/q}=$$

$$\frac{1}{2}C(\lambda,r)\left\{\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{r}\right)-1}\varphi^2(m)\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\psi^2(n)\right\}^{1/q}$$

证毕

定理5　设 $p_i,q_i,\{a_m\},\{b_n\}$ 和 $K(x,y)$ 如定理3所述，$A_m=\dfrac{1}{m}\sum_{s=1}^{m}a_s,B_n=\dfrac{1}{n}\sum_{t=1}^{n}b_t$，$\varphi$ 和 ψ 为定义在 $\mathbf{R}_+=[0,\infty)$ 上的非负凸、次可乘的可测实值函数，则

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn}{m+n}\cdot K(m,n)\varphi(A_m)\psi(B_n)\leqslant\frac{1}{2}C(\lambda,r)\left\{\sum_{m=1}^{\infty}m^{P\left(1-\frac{\lambda}{r}\right)-1}\widetilde{\varphi}^P(m)\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\widetilde{\psi}^q(n)\right\}^{1/q}$$

其中 $\widetilde{\varphi}(m)=\left\{\sum_{s=1}^{m}[(\varphi(a_s)]^2\right\}^{1/2}$ 和 $\widetilde{\psi}(n)=\left\{\sum_{t=1}^{n}[\psi(b_t)]^2\right\}^{1/2}$。

证明　由假设，应用Jensen不等式和Hölder不等式，得

$$\varphi(A_m)=\varphi\left(\frac{1}{m}\sum_{s=1}^{m}a_s\right)\leqslant\frac{1}{m}\sum_{s=1}^{m}[\varphi(a_s)]\leqslant\frac{1}{m}\cdot m^{1/2}\left\{\sum_{s=1}^{m}[\varphi(a_s)]^2\right\}^{1/2}$$

类似地，$\psi(B_n)\leqslant\frac{1}{n}\cdot n^{1/2}\left\{\sum_{t=1}^{n}[\psi(b_t)]^2\right\}^{1/2}$。应用 Young 不等式，得

$$\varphi(A_m)\psi(B_n)\leqslant\frac{m+n}{2mn}\left\{\sum_{s=1}^{m}[\varphi(a_s)]^2\right\}^{1/2}\left\{\sum_{t=1}^{n}[\psi(b_t)]^2\right\}^{1/2}\tag{4}$$

记 $\tilde{\varphi}(m)=\left\{\sum_{s=1}^{m}[\varphi(a_s)]^2\right\}^{1/2}$，$\tilde{\psi}(n)=\left\{\sum_{t=1}^{n}[\psi(b_t)]^2\right\}^{1/2}$。

(4)式两边乘以 $K(m,n)/(m+n)$，关于 m,n 求和，由 Hölder 不等式和引理 2，得

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn}{m+n}\cdot K(m,n)\varphi(A_m)\psi(B_n)\leqslant\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\left\{\sum_{s=1}^{m}[\varphi(a_s)]^2\right\}^{1/2}\left\{\sum_{t=1}^{n}[\psi(b_t)]^2\right\}^{1/2}\leqslant$$

$$\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\tilde{\varphi}(m)\tilde{\psi}(n)\leqslant\frac{1}{2}\left\{\sum_{m=1}^{\infty}v_1(\lambda,w,m)m^{p\left(1-\frac{\lambda}{r}\right)-1}\tilde{\varphi}^p(m)\right\}^{1/p}\left\{\sum_{n=1}^{\infty}v_2(\lambda,r,n)n^{q\left(1-\frac{\lambda}{w}\right)-1}\tilde{\psi}^q(n)\right\}^{1/q}<$$

$$\frac{1}{2}C(\lambda,r)\left\{\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{r}\right)-1}\tilde{\varphi}^p(m)\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\tilde{\psi}^q(n)\right\}^{1/q}$$

证毕

定理 6　若 $p_i,q_i,\{a_m\},\{b_n\},\{p_m\},\{q_n\},P_m,Q_n$ 和 $K(x,y)$ 如定理 4 所述，$A_m=\frac{1}{P_m}\sum_{s=1}^{m}p_sa_s$，$B_n=\frac{1}{Q_n}\sum_{t=1}^{n}q_tb_t$，$\varphi,\psi$ 为定义在 $\mathbf{R}_+=[0,\infty)$ 上的非负凸、次可乘的可测实值函数，则

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(m,n)P_mQ_n\varphi(A_m)\psi(B_n)}{m+n}\leqslant\frac{1}{2}C(\lambda,r)\left(\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{r}\right)-1}\varphi_2^p(m)\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\psi_2^q(n)\right\}^{1/q}$$

其中 $\varphi_2(m)=\left\{\sum_{s=1}^{m}[(p_s\varphi(a_s)]^2\right\}^{1/2}$ 和 $\psi_2(n)=\left(\sum_{t=1}^{n}[q_t\psi(b_t)]^2\right\}^{1/2}$。

证明　由假设，应用 Jensen 不等式和 Hölder，不等式，得

$$\varphi(A_m)=\varphi\left(\frac{1}{p_m}\sum_{s=1}^{m}p_sa_s\right)\leqslant\frac{1}{P_m}\sum_{s=1}^{m}[p_s\varphi(a_s)]\leqslant\frac{1}{P_m}.\sqrt{m}\left\{\sum_{s=1}^{m}[\varphi(p_sa_s)]^2\right\}^{1/2}$$

类似地，$\psi(B_n)\leqslant\frac{1}{Q_n}\cdot\sqrt{n}\left\{\sum_{t=1}^{n}[q_t\psi(b_t)]^2\right\}^{1/2}$。

应用 Young 不等式，得 $\varphi(A_m)\psi(B_n)\leqslant\frac{1}{P_mQ_n}\frac{m+n}{2}\left\{\sum_{s=1}^{m}[p_s\varphi(a_s)]^2\right\}^{1/2}\left\{\sum_{t=1}^{n}[q_t\psi(b_t)]^2\right\}^{1/2}$。记 $\varphi_2(m)=\left\{\sum_{s=1}^{m}[p_s\varphi(a_s)]^2\right\}^{1/2}$，$\psi_2(t)=\left\{\sum_{t=1}^{n}[q_t\psi(b_t)]^2\right\}^{1/2}$，余下的证明类似于定理 3，从略。　　证毕

3 一些应用的例子

推论 1　若 $\alpha=\beta=1,p,r>1,\frac{1}{p}+\frac{1}{q}=1,\frac{1}{r}+\frac{1}{w}=1,\mu>0,0<\lambda\leqslant\min\{r,w\}$，$K(x,y)=\frac{1}{(x^\mu+y^\mu)^{\lambda/\mu}}$，$a_m\geqslant0,b_n\geqslant0,A_m=\sum_{s=1}^{m}a_s,B_n=\sum_{t=1}^{n}b_t$，则

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{(m+n)(m^\mu n^\mu)^{\lambda/\mu}}\leqslant\frac{1}{2\mu}B\left(\frac{\lambda}{r\mu},\frac{\lambda}{w\mu}\right)\left\{\sum_{m=1}^{\infty}m^{p\left(1-\frac{\lambda}{r}\right)-1}\tilde{A}_m^p\right\}^{1/p}\times\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{w}\right)-1}\tilde{B}_n^q\right\}^{1/q}\tag{5}$$

除非 $\{a_m\}$ 或 $\{b_n\}$ 恒为 0，其中 $\tilde{A}_m=\left\{\sum_{s=1}^{m}a_s^2\right\}^{1/2}$ 和 $\tilde{B}_n=\left\{\sum_{t=1}^{n}b_t^2\right\}^{1/2}$。特别地，令 $\mu=\lambda=1$，有

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{(m+n)^2}\leqslant\frac{\pi}{2\sin(\pi/r)}\left\{\sum_{m=1}^{\infty}m^{\frac{p}{w}-1}\tilde{A}_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}\tilde{B}_n^q\right\}^{1/q}\tag{6}$$

证明　显然 $K(x,y)=\frac{1}{(x^\mu+y^\mu)^{\lambda/\mu}}$ 为 $(0,\infty)\times(0,\infty)$ 上的 $-\lambda$ 齐次非负可测函数，由定理 3 知，只要算得 $C(\lambda,r):=\int_0^\infty K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u$ 是仅与 λ,r 有关的正数即可，事实上，令 $v=u^\mu$，则

$$C(\lambda,r)=\int_0^\infty K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u=\int_0^\infty\frac{1}{(u^\mu+1)^{\lambda/\mu}}u^{\frac{\lambda}{r}-1}\mathrm{d}u=\frac{1}{\mu}\int_0^\infty\frac{1}{(v+1)^{\lambda/\mu}}v^{\frac{\lambda}{r\mu}-1}\mathrm{d}v=\frac{1}{\mu}B\left(\frac{\lambda}{r\mu},\frac{\lambda}{w\mu}\right)$$

可见(5)式成立。令 $\mu=\lambda=1$，得(6)式。　　证毕

推论 2　若 $\alpha=\beta=1,p,r>1,\frac{1}{p}+\frac{1}{q}=1,\frac{1}{r}+\frac{1}{w}=1,0<\mu\leqslant\lambda<\min\{r,w\}$，$K(x,y)=\frac{1}{(x+y)^{\lambda-\mu}(\max\{x,y\})^\mu}$，

$a_s \geqslant 0, b_t \geqslant 0, A_m = \sum_{s=1}^{m} a_s$ 和 $B_n = \sum_{t=1}^{n} b_t$,则

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m B_n}{(m+n)^{1+\lambda-\mu}(\max\{m,n\})^{\mu}} \leqslant \frac{1}{2}C(\lambda,r)\left\{\sum_{m=1}^{\infty} m^{p\left(1-\frac{\lambda}{r}\right)-1}\tilde{A}_m^{p}\right\}^{1/p} \times \left\{\sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{w}\right)-1}\tilde{B}_n^{q}\right\}^{1/q} \tag{7}$$

除非$\{a_m\}$或$\{b_n\}$恒为 0,其中 $C(\lambda,r) = \sum_{i=0}^{\infty}\binom{\alpha-\lambda}{i}\frac{(\lambda+2i)rw}{(\lambda+ri)(\lambda+wi)}$,$\tilde{A}_m = \left\{\sum_{s=1}^{m} a_s^2\right\}^{1/2}$ 和 $\tilde{B}_n = \left\{\sum_{t=1}^{n} b_t^2\right\}^{2}$。特别地,令 $\alpha=\lambda=1$,(7)式退化为

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m B_n}{(m+n)\max\{m,n\}} \leqslant \frac{rw}{2}\left\{\sum_{m=1}^{\infty} m^{\frac{p}{w}-1}\tilde{A}_m^{p}\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{\frac{q}{r}-1}\tilde{B}_n^{q}\right\}^{1/q} \tag{8}$$

证明 显然 $K(x,y)=\frac{1}{(x+y)^{\lambda-\mu}(\max\{x,y\})^{\mu}}$是$(0,\infty)\times(0,\infty)$上的$-\lambda$ 齐次非负可测函数,类似地,

$$C(\lambda,r)=\int_0^{\infty}K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u=\int_0^1(u+1)^{\mu-\lambda}\left(u^{\frac{\lambda}{r}-1}+u^{\frac{\lambda}{w}-1}\right)\mathrm{d}u=$$

$$\int_0^1\sum_{i=0}^{\infty}\binom{\mu-\lambda}{i}u^{i}\left(u^{\frac{\lambda}{r}-1}+u^{\frac{\lambda}{w}-1}\right)\mathrm{d}u=\sum_{i=0}^{\infty}\binom{\mu-\lambda}{i}\frac{(\lambda+2i)rw}{(\lambda+ri)(\lambda+wi)}$$

因此(7)式成立,令 $\alpha=\lambda=1$,得(8)式。 证毕

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Generalizations of Pachpatte's Discrete Inequality

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Abstract: The present paper is devoted to study the Pachpatte's inequality and the similar inequality, which is the variant of classical Hilbert inequality. By introducing the homogeneous function $K(x, y)$ of $-\lambda$ degree and two pairs of conjugate exponents (p, q), (r, s), $(1/p) + (1/p) = 1$, $(1/r) + (1/s) = 1$, by the proper identical transformation, and then using techniques of several classical inequalities (e. g. Hölder's inequality, Young's inequality and Jensen's inequality) and methods of real analysis to estimate the weight function, a series of Pachpatte's discrete inequalities and similar forms are obtained, which included a variety of inequality under the non-negative convex sub-multiplicative measurable real-valued function. This conclusion is the integrated use of the deduction of Hilbert's inequality and Pachpatte's inequality, putting non-conjugated exponents or only one pair of conjugate exponents of Pachpatte's inequality previously into the inequality of two pairs of conjugate exponents and multi-parameter, which unified the research of some literature, this makes the study of Pachpatte's inequality to rise to a higher level. As applications, we take two special functions of the homogeneous function $K(x, y)$, and some interesting inequalities are given.

Key words: Hölder's inequality; Young's inequality; Jensen's inequality.

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