运筹学与控制论

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一类不可微多目标规划的 Wolfe 型对偶*

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摘要:G-不变凸函数是一类新的广义凸函数,是 G-凸函数的推广。本文主要研究了一类带等式和不等式约束的目标函数带支撑函数的不可微多目标规划问题。首先,构造了该问题的 Wolfe 型对偶模型。其次,利用 G-Karush-Kuhn-Tucker 最优性必要条件,分别在 G-不变凸和 G-拉格朗日函数不变凸假设下证明了该问题及其对偶问题的弱对偶定理。最后,在适当条件下给出该问题及其对偶问题的强对偶和逆对偶定理及其证明。本文的结论更具一般性,将前人的相关结论推广到了非可微的情形。

关键词:多目标规划;不可微规划;G-不变凸;Wolfe 对偶

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凸性和广义凸性在最优化理论和应用中有较深远的影响,目前越来越多地被应用到数理经济、管理科学等领域。1981 年,Hanson 在文献[1]中提出不变凸函数的概念,自此各种新的不变凸函数的概念被相继提出[2-4]。2007 年,Antczak 在文献[5]中提出一类实值 G-不变凸函数。随后 Antczak 将它推广到向量情形,并且用它扩展了带等式和不等式的可微多目标规划的最优性条件和对偶[6-7]。2010 年,Ho Jung Kim 等人将文献[8]中的结论推广到了不可微的情形。

受以上文献的启发,本文主要研究了一类带等式和不等式约束的目标函数带支撑函数的不可微多目标规划问题,提出该问题的 Wlofe 对偶模型并证明了相关对偶定理。

1 预备知识

定义 1[9]　在多目标规划问题中,有如下规定:

$$\forall x=(x_1,x_2,\cdots,x_n)^{\mathrm T},y=(y_1,y_2,\cdots,y_n)^{\mathrm T}$$

$$x=y\Leftrightarrow x_i=y_i,i=1,2,\cdots,n;x<y\Leftrightarrow x_i<y_i,i=1,2,\cdots,n;$$

$$x\leqslant y\Leftrightarrow x_i\leqslant y_i,i=1,2,\cdots,n;x\leqslant y\Leftrightarrow x_i\leqslant y_i,x\neq y,n>1。$$

定义 2　函数 $f:\mathbf{R}\to\mathbf{R}$ 称为严格递增函数,当且仅当

$$\forall x,y\in\mathbf{R},x<y\Rightarrow f(x)<f(y)。$$

令 $f:=(f_1,f_2,\cdots,f_k):X\to\mathbf{R}^k$ 是定义在非空开集 $X\subset\mathbf{R}^n$ 上的向量值可微函数,k 为 X 在 f_i 作用下的像。

定义 3[6]　$f:X\to\mathbf{R}^k$ 定义在非空集 $X\subset\mathbf{R}^n$ 上可微向量值函数,$u\in X$,若存在一个可微向量值函数 $G_f=(G_{f_1},\cdots,G_{f_k}):\mathbf{R}\to\mathbf{R}^k$,其中 $G_{f_i}:I_{f_i}(X)\to\mathbf{R}$ 是严格增函数,存在 $\eta:X\times X\to\mathbf{R}^n$,使得对于 $\forall x\in X(x\neq u)$,$\forall i=1,\cdots,k$ 有

$$G_{f_i}(f_i(x))-G_{f_i}(f_i(u))-G'_{fi}(f_i(u))\nabla f_i(u)\eta(x,u)\geqslant 0(>)\tag{1}$$

则 f 称为在 u 关于 η 的(严格)G_f-不变凸。

注 1　类似地,定义(严格)G-不变凹函数,只需将定义中不等号改变到相反方向。

注 2　当对任意的 $a\in I_{f_i}(X)$,都有 $G_{f_i}(a)=a,i=1,2,\cdots,k$ 时,定义退化到向量值不变凸函数的定义。

定义 4[10]　C 为 $\mathbf{R}^n$ 中的一个紧凸集,C 的支撑函数定为

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$$s(x|C):=\max\{x^{\mathrm{T}}y:y\in C\}$$

凸且处处有限的支撑函数有次微分，即存在 $z\in\mathbf{R}^n$ 使得

$$s(y|C)\geqslant s(x|C)+z^{\mathrm{T}}(y-x),\forall y\in C$$

等价的有

$$z^{\mathrm{T}}x=s(x|C)$$

$s(x|C)$的次微分定义为

$$\partial s(x|C)=\{z\in C|z^{\mathrm{T}}x=s(x|C)\}$$

定义 5　可行点 $\bar{x}$ 称为(NMP)问题的弱 Pareto 解，若不存在 $x\in D$ 使得

$$G_{f(x)+x^{\mathrm{T}}\omega}f(x)+s(x|C)<G_{f(\bar{x})+(\bar{x})^{\mathrm{T}}\omega}f(\bar{x})+s(\bar{x}|C)。$$

定义 6　W 为 $\mathbf{R}^n$ 中一个给定的集合，其中由<或≦定义序关系。向量 $\bar{z}\in W$ 称为 W 中的 Pareto 解，如果在 W 中不存在 z 使得 $z\leqslant\bar{z}$；向量 $\bar{z}\in W$ 称为 W 中的弱 Pareto 解，如果在 W 中不存在 z 使得 $z\leqslant\bar{z}$。

命题 7　$\bar{z}$ 是多目标规划问题的可行解，令 $G_{f_i(\cdot)+(\cdot)^{\mathrm{T}}\omega_i},i=1,2,\cdots,k$ 是定义在 $I_{fi(\cdot)(\cdot)^{\mathrm{T}}}\omega_i(X)$上的连续实值严格增函数。进一步定义：

$$W=\{G_{f_1(\cdot)+(\cdot)^{\mathrm{T}}\omega_1}(f_1(x)+s(x|C_1)),\cdots,G_{f_k(\cdot)+(\cdot)^{\mathrm{T}}\omega_k}(f_k(x)+s(x|C_k)):x\in X\}\subset\mathbf{R}^k$$

$$\bar{z}=(G_{f_1(\cdot)+(\cdot)^{\mathrm{T}}\omega_1}(f_1(\bar{x})+s(\bar{x}|C_1)),\cdots,G_{f_k(\cdot)+(\cdot)^{\mathrm{T}}\omega_k}(f_k(\bar{x})+s(\bar{x}|C_k)))\in W$$

则 $\bar{x}$ 是多目标规划可行解中的弱 Pareto 解当且仅当对应的 $\bar{z}$ 是 W 中弱 Pareto 解。

定义 8　多目标规划问题被称为在 $\bar{x}\in D$ 满足 Kuhn-Tucker 约束规格，如果

$$C(D,\bar{x})=\{d\in\mathbf{R}^n:\nabla g_j(\bar{x})d\leqq 0,j\in J(\bar{x}),\nabla h_t(\bar{x})d=0,t\in T\}$$

本文研究一类带等式和不等式约束的目标函数带支撑函数的非可微多目标问题(NMP)：

$$\min G_{F_1}((f_1(x)+s(x|C_1)),\cdots,G_{F_k}(f_k(x)+s(x|C_k)))$$

$$\text{s. t.}\quad (G_{g_1}(g_1(x)),\cdots,G_{g_m}(g_m(x)))\leqq 0,$$

$$(G_{h_1}(h_1(x)),\cdots,G_{h_p}(h_p(x)))=0$$

其中，$f_i:X\to\mathbf{R},i\in I=\{1,2,\cdots,k\},g_j:X\to\mathbf{R},j\in J=\{1,2,\cdots,m\},h_t:X\to\mathbf{R},t\in T=\{1,2,\cdots,p\}$是定义在非空开集 $X\subset\mathbf{R}^n$ 上的可微函数，$G_{F_i},i\in I,G_{g_j},j\in J,G_{h_t},t\in T$ 是可微实值严格递增函数。

令 $D=\{x\in X:G_{g_j}(g_j(x))\leqq 0,j\in J,G_{h_t}(h_t(x))=0,t\in T\}$ 为(NMP)问题的可行解集，并且有 $F_i=f_i(\cdot)+(\cdot)^{\mathrm{T}}\omega_i$。进一步定义不等式约束函数在 $z\in D$ 的起作用集为 $J(z):=\{j\in J:G_{g_j}(g_j(z))=0\}$，目标函数指标集为 $I(z):=\{i\in I:\lambda_i>0\}$，其对应的拉格朗日乘子不等于零。

2 主要结论及其证明

定理 1　(G-Karush-Kuhn-Tucker 必要条件)[8] 设 $G_{F_i},i\in I$ 是定义在 $I_{F_i}(D)$上的可微实值严格增函数，$G_{g_j},j\in J$ 是定义在 $I_{g_j}(D)$上的可微实值严格增函数，$G_{h_t},t\in T$ 是定义在 $I_{h_t}(D)$上的可微实值严格增函数，$G_{h_t},t\in T$线性无关的，$F_i=f_i(\cdot)+(\cdot)^{\mathrm{T}}\omega_i$。进一步，假设存在 $z^*\in\mathbf{R}^n$，使得

$$\langle G'_{g_j}(g_j(\bar{x})\nabla g_j(\bar{x}),z^*\rangle<0,j\in J(\bar{x}),\langle G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}),z^*\rangle=0,t=1,2,\cdots,p$$

若 $\bar{x}\in D$ 是(NMP)问题的一个弱 Pareto 最优点，则存在 $\lambda\in\mathbf{R}^k_+,\xi\in\mathbf{R}^m_+,\mu\in\mathbf{R}^p,\omega_i\in C_i,i=1,2,\cdots,k$ 使得

$$\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(\bar{x})+\bar{x}^{\mathrm{T}}\omega_i)(\nabla f_i(\bar{x})+\omega_i)+\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})=0 \tag{2}$$

$$\xi_j G_{g_j}(g_j(\bar{x}))=0,j\in J \tag{3}$$

$$\langle\omega_i,\bar{x}\rangle=s(\bar{x}|C_i),i=1,2,\cdots,k \tag{4}$$

$$\lambda\geqslant 0,\sum_{i=1}^{k}\lambda_i=1,\xi\geqq 0 \tag{5}$$

给出问题(NMP)的 Wolfe 对偶模型(NWD)为

$$\Big(G_{F_1}(f_1(y)+s(y\mid C_1))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))\nabla g_j(y)+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\nabla h_t(y),\cdots,$$

$$G_{F_k}(f_k(y)+s(y\mid C_k))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))\nabla g_j(y)+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\nabla h_t(y)\Big)\rightarrow$$

$$\max\Big[\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(y)+s(y\mid C_i))(\nabla f_i(y)+\omega_i)+\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(y))\nabla g_j(y)+$$

$$\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\Big]\cdot\eta(x,y)\geqq 0,\forall x\in D,$$

$$y\in X,\lambda\in\mathbf{R}^n,\lambda\geqslant 0,\lambda e=1,\xi\in\mathbf{R}^m,\xi\geqq 0,\mu\in\mathbf{R}^p$$

其中 $e=(1,1,\cdots,1)\in\mathbf{R}^k$，设 G_{f_i}，$i\in I$ 是定义在 $I_{F_i}(X)$上的可微实值严格增函数，G_{g_i}，$j\in J$ 是定义在 $I_{g_i}(X)$上的可微实值严格增函数，G_{h_t}，$t\in T$ 是定义在 $I_{h_t}(X)$上的可微实值严格增函数，$F_i=f_i(\cdot)+(\cdot)^{\mathrm{T}}\omega_i$。

令 W 为(NWD)问题的可行解集，$pr_x\widetilde{W}:=\{y\in X:(y,\lambda,\zeta,\mu)\in\widetilde{W}\}$。

为了证明多目标规划问题的 Wolfe 对偶模型，作者定义向量 G-拉格朗日函数 $L_G:X\times\mathbf{R}^k\times\mathbf{R}^m\times\mathbf{R}^p\rightarrow\mathbf{R}^k$ 如下：

$$L_G(y,\lambda,\xi,\mu)=diag\,\lambda\,(G_{F_1}(f_1(y)+s(y\mid C_1)),\cdots,G_{F_k}(f_k(y)+s(y\mid C_k)))^{\mathrm{T}}+$$

$$\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))e+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))e=$$

$$\Big(\lambda_1 G_{F_1}(f_1(y)+s(y\mid C_1))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y)),\cdots,$$

$$\lambda_k G_{F_k}(f_k(y)+s(y\mid C_k))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\Big)$$

其中 $diag\,\lambda=\begin{bmatrix}\lambda_1 & 0 & 0 & \cdots & 0\\ 0 & \lambda_2 & 0 & \cdots & 0\\ 0 & & \ddots & & \vdots\\ \vdots & & & \ddots & 0\\ 0 & \cdots & \cdots & 0 & \lambda_k\end{bmatrix}$。 证毕

定理 2 (弱对偶)x 和(y,λ,ζ,μ)分别是(NMP)和(NWD)的任意可行解。假设 F 在 $y\in pr_X\widetilde{W}$ 关于 η 是 G_F-不变凸，g 在 $y\in pr_X\widetilde{W}$ 关于 η 是 G_g-不变凸，h_t，$t\in T^+(y)$在 $y\in pr_X\widetilde{W}$ 关于 η 是 G_{h_t}-不变凸，h_t，$t\in T^-(y)$在 $y\in pr_X\widetilde{W}$ 关于 η 是 G_{h_t}-不变凹，$G_{g_j}(0)=0$，$j\in J$，$G_{h_t}(0)=0$，$t\in T^+(y)\cup T^-(y)$。则

$$(G_{F_1}(f_1(x)+s(x\mid C_1)),\cdots,G_{F_k}(f_k(x)+s(x\mid C_k)))\not<\Big(G_{F_1}(f_1(y)+s(y\mid C_1))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+$$

$$\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y)),\cdots,G_{F_k}(f_k(y)+s(y\mid C_k))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\Big)。$$

证明 反证法，假设

$$(G_{F_1}(f_1(x)+s(x\mid C_1)),\cdots,G_{F_k}(f_k(x)+s(x\mid C_k)))<\Big(G_{F_1}(f_1(y)+s(y\mid C_1))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+$$

$$\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y)),\cdots,G_{F_k}(f_k(y)+s(y\mid C_k))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\Big)$$

则对于任意的 $i\in I$，有

$$G_{F_i}(f_i(x)+s(x\mid C_i))-G_{F_i}(f_i(y)+s(y\mid C_i))<\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y)+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))$$

因为 $\lambda\geqslant 0$，则上式有

$$\sum_{i=1}^{k}\lambda_i G_{F_i}(f_i(x)+s(x\mid C_i))-\sum_{i=1}^{k}\lambda_i G_{F_i}(f_i(y)+s(y\mid C_i))<\sum_{i=1}^{k}\lambda_i\Big[\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\Big]$$

由 (y,λ,ζ,μ) 是(NWD) 的可行解，有 $\sum_{i=1}^{k}\lambda_i=1$，上面的不等式蕴含着

$$\sum_{i=1}^{k}\lambda_i G_{F_i}(f_i(x)+s(x\mid C_i)-\sum_{i=1}^{k}\lambda_i G_{F_i}(f_i(y)+s(y\mid C_i))<\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y)+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y)) \tag{6}$$

因为 $x\in D,G_{g_j},j\in J,G_{h_t},t\in T$ 在他们的定义域上是严格增函数,$G_{g_j}(0)=0,j\in J,G_{h_t}(0)=0,t\in T^+(y)\cup T^-(y)$,所以有

$$G_{g_j}(g_j(x))\leqq 0,j\in J,G_{h_t}(h_t(x))=0,t\in T^+(x)\cup T^-(x)$$

因此,由(y,λ,ζ,μ)的可行性,有

$$\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(x)+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(x))\leqq 0 \tag{7}$$

因为 F 在 $y\in pr_X\widetilde{W}$ 关于 η 是 G_F-不变凸,g 在 $y\in pr_X\widetilde{W}$ 关于 η 是 G_g-不变凸,$h_t,t\in T^+(y)$在 $y\in pr_X\widetilde{W}$ 关于 η 是 G_{h_t}-不变凸,$h_t,t\in T^-(y)$在 $y\in pr_X\widetilde{W}$ 关于 η 是 G_{h_t}-不变凹,由定义,有如下式子成立:

$$G_{F_i}(f_i(\tilde{x})+s(\tilde{x}|C_i))-G_{F_i}(f_i(y)+s(y|C_i))-$$
$$G'_{F_i}(f_i(y)+s(y|C_i))(\nabla f_i(y)+\omega_i)\eta(\tilde{x},y\geqq 0,i\in I$$
$$G_{g_i}(g_j(\tilde{x}))-G_{g_j}(g_j(y))-G'_{g_j}(g_j(y))\nabla g_j(\bar{y})\eta(\tilde{x},y)\geqq 0,j\in J$$
$$G_{h_t}(h_t(\tilde{x})-G_{h_t}(y))-G'_{h_t}(h_t(y))\nabla h_t(y)\eta(\tilde{x},y)\geqq 0,t\in T^+(y)$$
$$G_{h_t}(h_t(\tilde{x}))-G_{h_t}(h_t(y))-G'_{h_t}(h_t(y))\nabla h_t(y)\eta(\tilde{x},y)\leqq 0,t\in T^-(y)$$

由$(y,\bar{\lambda},\bar{\zeta},\bar{\mu})$在(NWD)的可行性,有

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i G_{F_i}(f_i(x)+s(x\mid C_i))-\sum_{i=1}^{k}\lambda_i G_{F_i}(f_i(y)+s(y\mid C_i))\geqq\\&\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(y)+s(y\mid C_i))(\nabla f_i(y)+\omega_i)\eta(x,y),i\in I\end{aligned} \tag{8}$$

$$\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(x))-\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))\geqq\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(y))\nabla gj(y)\eta(x,y),j\in J \tag{9}$$

$$\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(x))-\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\geqq\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\eta(x,y),j\in T \tag{10}$$

由反设,(8)式和$(y,\bar{\lambda},\bar{\zeta},\bar{\mu})$在(NWD)的可行性,可得

$$\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(x))-\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))>\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(y))+s(y\mid C_i)(\nabla f_i(y)+\omega_i)\eta(x,y),i\in I \tag{11}$$

把(9)～(11)式左右两边分别相加得

$$\begin{aligned}&\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(x))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(x))\geqq\Big[\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(y)+s(y\mid C_i))(\nabla f_i(y)+\omega_i)+\\&\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(y))\nabla g_j(y)+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\Big]\eta(x,y)\end{aligned}$$

与(7)式联立,可得如下不等式

$$\Big[\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(y)+s(y\mid C_i))(\nabla f_i(y)+\omega_i)+\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(y))\nabla g_j(y)+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\Big]\cdot\eta(x,y)<0$$

与(y,λ,ζ,μ)的可行性矛盾,得证。　　证毕

下面,在 G-拉格朗日函数不变凸的条件下证明 G-弱对偶定理。

定理 3　(弱对偶)x 和(y,λ,ζ,μ)分别是(NMP)和(NWD)的任意可行解,假设 G-拉格朗日函数 L_G 在 $y\in pr_X\widetilde{W}$ 关于 η 不变凸,$G_{g_j}(0)=0,j\in J,G_{h_t}(0)=0,t\in T^+(y)\cup T^-(y)$,则

$$\begin{aligned}&(G_{F_1}(f_1(x)+s(x\mid C_1)),\cdots,G_{F_k}(f_k(x)+s(x\mid C_k)))\not< \Big(G_{F_1}(f_1(y)+s(y\mid C_1))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\\&\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y)),\cdots,G_{F_k}(f_k(y)+s(y\mid C_k))+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\Big)\end{aligned}$$ 。

证明　反证法。假设

$$(G_{F_1}(f_1(x)+s(x\mid C_1)),\cdots,G_{F_k}(f_k(x)+s(x\mid C_k)))<\Big(G_{F_1}(f_1(y)+s(y\mid C_1))+\sum_{j=1}^{m}\xi_jG_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_tG_{h_t}(h_t(y)),\cdots,G_{F_k}(f_k(y)+s(y\mid C_k))+\sum_{j=1}^{m}\xi_jG_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_tG_{h_t}(h_t(y))\Big)$$

与定理1相似,可以得到(6)、(7)式,两式相加可得

$$\sum_{i=1}^{k}\lambda_iG_{F_i}(f_i(x)+s(x\mid C_i))+\sum_{j=1}^{m}\xi_jG_{g_j}(g_j(x))+\sum_{t=1}^{p}\mu_tG_{h_t}(h_t(x))<\sum_{i=1}^{k}\lambda_iG_{F_i}(f_i(y)+s(y\mid C_i))+\sum_{j=1}^{m}\xi_jG_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_tG_{h_t}(h_t(y)) \tag{12}$$

因为 L_G 是关于 η 的不变凸函数,所以有

$$L_G(x,\lambda,\zeta,\mu)-L_G(y,\lambda,\zeta,\mu)\geqq\nabla L_G(y,\lambda,\zeta,\mu)\eta(x,y)$$

因此由 L_G 的定义,对于任意的 $i=1,\cdots,k$,有

$$\lambda_iG_{F_i}(f_i(x)+s(x\mid C_i))+\Big[\sum_{j=1}^{m}\xi_jG_{g_j}(g_j(x))+\sum_{t=1}^{p}\mu_tG_{h_t}(h_t(x))\Big]-\lambda_iG_{F_i}(f_i(y)+s(y\mid C_i))-\Big[\sum_{j=1}^{m}\xi_jG_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_tG_{h_t}(h_t(y))\Big]\geqq\Big[\lambda_iG'_{F_i}(f_i(y)+s(y\mid C_i))(\nabla f_i(y)+\omega_i)+\sum_{j=1}^{m}\xi_jG'_{g_j}(g_j(y))\nabla g_j(y)+\sum_{t=1}^{p}\mu_tG'_{h_t}(h_t(y))\nabla h_t(y)\Big]\eta(x,y)$$

把上述不等式的左右两边相加,再使用 $\sum_{i=1}^{k}\lambda_i=1$,可以得到

$$\sum_{i=1}^{k}\lambda_iG_{F_i}(f_i(x)+s(x\mid C_i))+\Big[\sum_{j=1}^{m}\xi_jG_{g_j}(g_j(x))+\sum_{t=1}^{p}\mu_tG_{h_t}(h_t(x))\Big]-\sum_{i=1}^{k}\lambda_iG_{F_i}(f_i(y)+s(y\mid C_i))-\Big[\sum_{j=1}^{m}\xi_jG_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_tG_{h_t}(h_t(y))\Big]\geqq\Big[\sum_{i=1}^{k}\lambda_iG'_{F_i}(f_i(y)+s(y\mid C_i))(\nabla f_i(y)+\omega_i)+\sum_{j=1}^{m}\xi_jG'_{g_j}(g_j(y))\nabla g_j(y)+\sum_{t=1}^{p}\mu_tG'_{h_t}(h_t(y))\nabla h_t(y)\Big]\eta(x,y) \tag{13}$$

由(12)、(13)式,得

$$\Big[\sum_{i=1}^{k}\lambda_iG'_{F_i}(f_i(y))+s(y\mid C_i))(\nabla f_i(y)+\omega_i)+\sum_{j=1}^{m}\zeta_jG'_{g_i}(g_j(y))\nabla g_j(y)+\sum_{i=1}^{p}\mu_tG'_{h_t}(h_t(y))\nabla h_t(y)\Big]\eta(x,y)<0$$

与(y,λ,ζ,μ)的可行性矛盾,得证。 证毕

定理4 (强对偶)$\bar{x}$是(NMP)的一个弱Pareto解,假设$\bar{x}$满足Kuhn-Tucker约束规格,则存在$\bar{\lambda}\in\mathbf{R}_+^k$,$\bar{\xi}\in\mathbf{R}_+^m$,$\bar{\mu}\in\mathbf{R}^p$,使得$(\bar{x},\bar{\lambda},\bar{\zeta},\bar{\mu})$是(NWD)的可行解且(NMP)和(NWD)目标函数值相等。若定理1成立,则$(\bar{x},\bar{\lambda},\bar{\zeta},\bar{\mu})$是(NWD)的极大点。

定理5 (逆对偶)$(\bar{x},\bar{\lambda},\bar{\zeta},\bar{\mu})$是(NWD)的弱Pareto解且$\bar{y}\in D$,假设$G$-拉格朗日函数$L_G$在$y\in pr_X\widehat{W}$关于$\eta$是(严格)不变凸的,则$\bar{y}$是(NMP)的弱Pareto最优。

证明 反证法。假设$\bar{y}$不是(NMP)的弱Pareto解,即存在$\tilde{x}\in D$使得

$$f(\tilde{x}+s(\tilde{x}|C)<\tilde{f}+s(\tilde{y}|C)$$

由 G_{F_i}，$i\in I$ 是上的严格增函数，得

$$G_{F_i}(f_i(\tilde{x})+s(\tilde{x}|C_i))<G_{F_i}(f_i(\bar{y})+s(\bar{y}|C_i)),i\in I \tag{14}$$

$(\bar{x},\bar{\lambda},\bar{\zeta},\bar{\mu})$是(NWD)弱 Pareto 解，由 G-Karush-Kuhn-Tucker 必要条件(3)和 $\tilde{x}\in D$ 有

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{x}))\leqq\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{y})),j\in J \tag{15}$$

又因为 $\bar{y}\in D$ 和 $\bar{x}\in D$，有

$$\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{x}))-\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{y})) \tag{16}$$

由(14)～(16)式，对任意的 $i=1,\cdots,k$

$$G_{F_i}(f_i(\tilde{x})+s(\tilde{x}\mid C_i))+\Big[\sum_{j=1}^{m}\bar{\zeta}_j G_{g_j}(g_j(\tilde{x}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{x}))\Big]<$$
$$G_{F_i}(f_i(\bar{y})+s(\bar{y}\mid C_i))+\Big[\sum_{j=1}^{m}\bar{\zeta}_j G_{g_j}(g_j(\bar{y}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{y}))\Big]$$

因为 $\bar{\lambda}_i\geqslant 0,i\in I$，则有

$$\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\tilde{x})+s(\tilde{x}\mid C_i))+\sum_{i=1}^{k}\bar{\lambda}_i\Big[\sum_{j=1}^{m}\bar{\zeta}_j G_{g_j}(g_j(\tilde{x}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{x}))\Big]<$$
$$\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\bar{y})+s(\bar{y}\mid C_i))+\sum_{i=1}^{k}\bar{\lambda}_i\Big[\sum_{j=1}^{m}\bar{\zeta}_j G_{g_j}(g_j(\bar{y}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{y}))\Big]$$

由$(\bar{x},\bar{\lambda},\bar{\zeta},\bar{\mu})$在(NWD)的可行性，有 $\sum_{i=1}^{k}\bar{\lambda}_i=1$，所以上面不等式蕴含

$$\begin{aligned}&\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\tilde{x})+s(\tilde{x}\mid C_i))+\Big[\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{x}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{x}))\Big]<\\&\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\bar{y})+s(\bar{y}\mid C_i))+\Big[\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{y}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{y}))\Big]\end{aligned} \tag{17}$$

因为 L_G 是关于 η 的不变凸函数，所以由 G-拉格朗日的定义，对任意的 $i=1,\cdots,k$ 有

$$\bar{\lambda}_i G_{F_i}(f_i(\tilde{x})+s(\bar{x}\mid C_i))+\Big[\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{x})+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{x}))\Big]-$$
$$\bar{\lambda}_i G_{F_i}(f_i(\bar{y})+s(\bar{y}\mid C_i))-\Big[\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{y}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{y}))\Big]\geqq$$
$$[\bar{\lambda}_i G'_{F_i}(f_i(\bar{y})+s(\bar{y}\mid C_i))(\nabla f_i(\bar{y})+\omega_i)+\sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{y}))\nabla g_j(\bar{y})+$$
$$\sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{y}))\nabla h_t(\bar{y})\Big]\eta(\tilde{x},\bar{y})$$

把上述不等式左右两边分别相加，使用 $\sum_{i=1}^{k}\bar{\lambda}_i=1$，可得

$$\begin{aligned}&\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\tilde{x})+s(\tilde{x}\mid C_i))+\Big[\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{x}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{x}))\Big]-\\&\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\bar{y})+s(\bar{y}\mid C_i))-\Big[\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{y}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{y}))\Big]\geqq\\&\Big[\sum_{i=1}^{k}\bar{\lambda}_i G'_{F_i}(f_i(\bar{y})+s(\bar{y}\mid C_i))(\nabla f_i(\bar{y})+\omega_i)+\sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{y}))\nabla g_j(\bar{y})+\\&\sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{y}))\nabla h_t(\bar{y})\Big]\eta(\tilde{x},\bar{y})\end{aligned} \tag{18}$$

由(17)和(18)式，获得如下不等式

$$\Big[\sum_{i=1}^{k}\bar{\lambda}_i G'_{F_i}(f_i(\bar{y})+s(\bar{y}\mid C_i))(\nabla f_i(\bar{y})+\omega_i)+\sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{y}))\nabla g_j(y)+\sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{y}))\nabla h_t(\bar{y})\Big]\eta(\tilde{x},\bar{y})<0。$$

这和对偶问题(NWD)的约束条件矛盾,得证 证毕

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Operations Research and Cybernetics

Wolfe Duality for a Class of Nondifferentiable Multiobjective Programming

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Abstract: A G-invex function is a class of generalized convex functions. It is a generalization of the G-convex functions. In this paper, a class of nondifferentiable multiobjective programs with both inequality and equality constrains in which every component of the objective function contains a term involving the support function of a compact convex set are considered. Wlofe type dual problem is formulated firstly. Furthermore, we use G-Karush-Kuhn-Tucker necessary optimality conditions to establish weak duality theorems relating the problem and the dual problems under G-invex assumption and invex assumption of G-lagrange function respectively. In the final, strong duality theorem and converse duality theorem are established under suitable conditions. The work generalized some related results to the nondifferentiable case.

Key words: multiobjective programming; nondifferentiable programming; G-invex function;Wolfe duality

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