DOI:10.11721/cqnuj20140426

约束阻尼板的解析法和改进传递矩阵法*

万浩川[1,2]，郑　玲[1]，李以农[1]

(1. 重庆大学 机械传动国家重点实验室，重庆 400044；2. 乐山职业技术学院，四川 乐山 614000)

摘要：基于 Kirchhoff 假设和 Kerwin 假设，建立了三层约束阻尼板的振动方程。分别采用解析法和改进的传递矩阵法求解了方程，算例表明本文方法求解精度可靠。传递矩阵法边界条件适应性更强，改进的传递矩阵法通过引入一个关联矩阵，状态向量一阶导数的求解更加简单，这也简化了传递矩阵的求解，且振动方程自由度越多其优势越明显。本文计算方法可用于多夹层阻尼结构的求解及振动分析。

关键词：约束阻尼板；解析法；传递矩阵法；关联矩阵

中图分类号：O321　　**文献标志码**：A　　**文章编号**：1672-6693(2014)04-0131-05

约束阻尼层是一种夹层型结构，又称为夹层阻尼结构或剪切阻尼结构，广泛应用于航空发动机、风机叶片等薄壁结构的振动抑制中[1]。它是在需要减振的基体结构(基层)表面粘贴一层粘弹性阻尼材料(粘弹性阻尼层)，然后粘贴弹性金属层(约束层)，阻尼层和约束层交替布置，就形成了多夹层阻尼结构。当阻尼层随基层一起产生弯曲振动而使阻尼层产生拉压变形时，外层的约束层将会抑制阻尼层的拉压变形。由于阻尼层和基层接触表面所产生的拉压变形不同于和约束层接触表面所产生的拉压变形，因而阻尼材料内部产生剪切变形，又由于粘弹性阻尼材料具有很好的阻尼特性，能大量地耗散振动能量，起到很好的减振效果。

约束阻尼层结构可有三层、五层或更多层，三层约束阻尼结构的分析和计算是多夹层结构分析计算的基础。目前关于约束阻尼结构的研究多集中于三层结构，He[2]分析了附加阻尼材料的板的弯曲振动；Cupia[3]和 Wang[4]分别分析了约束阻尼矩形板和圆板的振动模态。钱振东[5]考虑了附加部分对原结构运动的相对性和阻尼层的横向剪切效应，据此推导了约束阻尼层板的运动方程和边界条件；最后分析了简支矩形板的固有振动。李恩奇等人[6-7]采用传递矩阵的方法分析了其它边界条件矩形板的固有振动，分别采用复常量和复变量模型求出了固有频率和结构的损耗因子。本文选用改进传递矩阵法[8]，在边界条件向量和状态向量之间引入一个关联矩阵，大大简化了传递矩阵的求解。采用解析和改进的传递矩阵法两种方法对约束阻尼板进行了求解，并对比了两种方法的优缺点。

1 约束阻尼板振动方程

典型的约束阻尼板如图 1 所示，由基层(1)、粘弹性阻尼层(2)和约束层(3)组成。图 2 为几何变形关系图，设基层和约束层厚度分别为 h_1、h_3，弹性模量分别为 E_1、E_3，泊松比分别为 μ_1、μ_3；阻尼层厚度为 h_2，复剪切模量为 G_2^*，板长为 a，宽为 b。

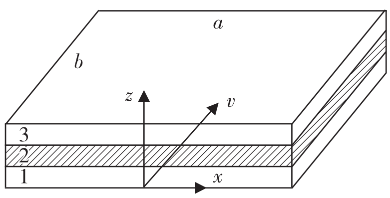

图 1　约束阻尼板示意图

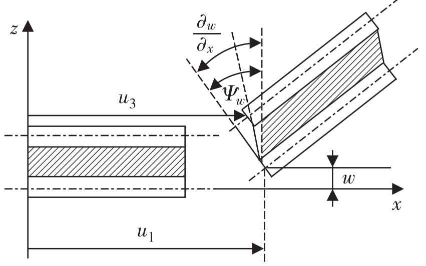

图 2　约束阻尼板变形图

对约束阻尼板做以下基本假设：1)三层纵向位移(挠度)相同；2)各层之间没有滑移，层间位移连续；3)基层和约束层采用克希霍夫(Kerchhoff)假设；4)只考虑粘弹层的剪切效应，忽略其纵向刚度(Kerwin 假设)；5)忽略转动惯量的影响。

* 收稿日期：2013-04-13　　修回日期：2013-12-09　　网络出版时间：2014-7-3 23:03
资助项目：四川省教育厅科研项目(No. 13ZB0351)；四川省科技厅应用基础计划(No. 2014JY0061)
作者简介：万浩川，男，副教授，博士研究生，研究方向为机械结构动力学、振动控制，E-mail：wanhaochuan@163.com
网络出版地址：http://www.cnki.net/kcms/detail/50.1165.N.20140703.2303.026.html

设基板位移向量 $x_1=\{u_1\ v_1\ w\}^{\mathrm{T}}$，其中 $u_1 v_1 w$ 分别为中性面在 x、y、z 三方向上的位移。基层内力为：

$$\begin{cases} N_{x1}=K_1\left(X\dfrac{\partial u_1}{\partial x}+\mu_1\dfrac{\partial v_1}{\partial y}\right) \\ N_{y1}=K_1\left(\mu_1\dfrac{\partial u_1}{\partial x}+\dfrac{\partial v_1}{\partial y}\right) \\ N_{xy1}=K_1\dfrac{1-\mu_1}{2}\left(\dfrac{\partial u_1}{\partial y}+\dfrac{\partial v_1}{\partial x}\right) \\ M_{x1}=D_1\left(\dfrac{\partial^2 w}{\partial x^2}+\mu_1\dfrac{\partial^2 w}{\partial y^2}\right) \\ M_{y1}=D_1\left(\mu_1\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\right) \\ M_{xy1}=D_1(1-\mu_1)\dfrac{\partial^2 w}{\partial x\partial y} \end{cases} \tag{1}$$

图2是粘弹性层几何变形关系图，粘弹性层应变列向量 $\gamma_2=\{\gamma_{xz}\gamma_{yz}\}^T$ 可表示为：$\gamma_{xz}=\left(u_3-u_1-h\dfrac{\partial w}{\partial x}\right)/h_2$。$\gamma_{xz}=\left(v_3-v_1-h\dfrac{\partial w}{\partial x}\right)/h_2$。其本构关系采用复常量模型，则应力向量 $\tau_2=\{\tau_{xz}\tau_{yz}\}^T$，有 $\tau_2=G_2*\gamma_2$。

$$\begin{cases} \tau_{xz}=\dfrac{G_2{}^*}{h_2}\left(u_3-u_1-h\dfrac{\partial w}{\partial x}\right) \\ \tau_{yz}=\dfrac{G_2{}^*}{h_2}\left(v_3-v_1-h\dfrac{\partial w}{\partial x}\right) \end{cases} \tag{2}$$

在 x、y、z 建立动-静力平衡方程，得式(3)：

$$\begin{cases} \dfrac{\partial N_{x1}}{\partial x}+\dfrac{\partial N_{xy1}}{\partial y}+\tau_{xz}=m_1\dfrac{\partial^2 u_1}{\partial t^2} \\ \dfrac{\partial N_{y1}}{\partial y}+\dfrac{\partial N_{xy1}}{\partial x}+\tau_{yz}=m_1\dfrac{\partial^2 v_1}{\partial t^2} \\ \dfrac{\partial Q_{x1}}{\partial x}+\dfrac{\partial Q_{y1}}{\partial y}=m_1\dfrac{\partial^2 w}{\partial t^2} \end{cases} \tag{3}$$

在 x、y 轴方向建立力矩平衡方程，得式(4)：

$$\begin{cases} \dfrac{\partial M_{xy1}}{\partial y}+\dfrac{\partial M_{x1}}{\partial x}-Q_{x1}-\dfrac{h_1}{2}\tau_{xz}=0 \\ \dfrac{\partial M_{xy1}}{\partial x}+\dfrac{\partial M_{y1}}{\partial y}-Q_{y1}-\dfrac{h_1}{2}\tau_{yz}=0 \end{cases} \tag{4}$$

由方程(4)求得 Q_{x1}、Q_{y1} 并代入(3)式，可得(5)：

$$\begin{cases} \dfrac{\partial N_{x1}}{\partial x}+\dfrac{\partial N_{xy1}}{\partial y}+\tau_{xz}=m_1\dfrac{\partial^2 u_1}{\partial t^2} \\ \dfrac{\partial N_{y1}}{\partial y}+\dfrac{\partial N_{xy1}}{\partial x}+\tau_{yz}=m_1\dfrac{\partial^2 v_1}{\partial t^2} \\ \dfrac{\partial M_{x1}^2}{\partial x^2}+\dfrac{\partial M_{y1}^2}{\partial y^2}+2\dfrac{\partial M_{xy1}^2}{\partial x\partial y} \\ -\dfrac{1}{2}h_1\left(\dfrac{\partial \tau_{xz}}{\partial_x}+\dfrac{\partial \tau_{yz}}{\partial_y}\right)=m_1\dfrac{\partial^2 w}{\partial t^2} \end{cases} \tag{5}$$

$$-h_2\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}\right)=m_2\frac{\partial^2 w}{\partial t^2} \tag{6}$$

对于约束层，受力情况和基层相同，只是 τ_{xz}、τ_{yz} 方向和基层相反，将方程(3)中的下标由“1”改为“3”，即可得约束层平衡方程(7)：

$$\begin{cases} \dfrac{\partial N_{x3}}{\partial x}+\dfrac{\partial N_{xy3}}{\partial y}-\tau_{xz}=m_3\dfrac{\partial^2 u_3}{\partial t^2} \\ \dfrac{\partial N_{y3}}{\partial y}+\dfrac{\partial N_{xy3}}{\partial x}-\tau_{yz}=m_3\dfrac{\partial^2 v_3}{\partial t^2} \\ \dfrac{\partial^2 M_{x3}}{\partial x^2}+\dfrac{\partial^2 M_{y3}}{\partial y^2}+2\dfrac{\partial^2 M_{xy3}}{\partial x\partial y} \\ -\dfrac{1}{2}h_3\left(\dfrac{\partial \tau_{xz}}{\partial_x}+\dfrac{\partial \tau_{yz}}{\partial_y}\right)=m_3\dfrac{\partial^2 w}{\partial t^2} \end{cases} \tag{7}$$

综合三层平衡方程(5)、(6)、(7)，可得约束阻尼板平衡方程(8)：对于粘弹性层，平衡方程为：

$$\begin{cases} \dfrac{\partial N_{x1}}{\partial x}+\dfrac{\partial N_{xy1}}{\partial y}+\tau_{xz}=m_1\dfrac{\partial^2 u_1}{\partial t^2}, \\ \dfrac{\partial N_{y1}}{\partial y}+\dfrac{\partial N_{xy1}}{\partial x}+\tau_{yz}=m_1\dfrac{\partial^2 v_1}{\partial t^2} \\ \dfrac{\partial N_{x3}}{\partial x}+\dfrac{\partial N_{xy3}}{\partial y}-\tau_{xz}=m_3\dfrac{\partial^2 u_3}{\partial t^2}, \\ \dfrac{\partial N_{y3}}{\partial y}+\dfrac{\partial N_{xy3}}{\partial x}-\tau_{yz}=m_3\dfrac{\partial^2 v_3}{\partial t^2} \\ \dfrac{\partial^2 M_{x1}}{\partial x^2}+\dfrac{\partial^2 M_{y1}}{\partial y^2}+2\dfrac{\partial^2 M_{xy1}}{\partial x\partial y}+\dfrac{\partial^2 M_{x3}}{\partial x^2}+\dfrac{\partial^2 M_{y3}}{\partial y^2} \\ +2\dfrac{\partial^2 M_{xy3}}{\partial x\partial y}-h\left(\dfrac{\partial \tau_{xz}}{\partial_x}+\dfrac{\partial \tau_{yz}}{\partial_y}\right)=m\dfrac{\partial^2 w}{\partial t^2} \end{cases} \tag{8}$$

将内力表达式(1)、(2)代入方程(8)，并整理可得约束阻尼板平衡方程：

$$\begin{cases} K_1\left(\dfrac{\partial^2 u_1}{\partial x^2}+\dfrac{1+\mu_1}{2}\dfrac{\partial^2 v_1}{\partial x\partial y}+\dfrac{1-\mu_1}{2}\dfrac{\partial^2 u_1}{\partial y^2}\right)+\dfrac{G_2{}^*}{h_2}\left(u_3-u_1-h\dfrac{\partial w}{\partial x}\right)=m_1\dfrac{\partial^2 u_1}{\partial t^2} \\ K_1\left(\dfrac{\partial^2 v_1}{\partial y^2}+\dfrac{1+\mu_1}{2}\dfrac{\partial^2 u_1}{\partial x\partial y}+\dfrac{1-\mu_1}{2}\dfrac{\partial^2 v_1}{\partial x^2}\right)+\dfrac{G_2{}^*}{h_2}\left(v_3-v_1-h\dfrac{\partial w}{\partial y}\right)=m_1\dfrac{\partial^2 v_1}{\partial t^2} \\ K_3\left(\dfrac{\partial^2 u_3}{\partial x^2}+\dfrac{1+\mu_3}{2}\dfrac{\partial^2 v_3}{\partial x\partial y}+\dfrac{1-\mu_3}{2}\dfrac{\partial^2 u_3}{\partial y^2}\right)-\dfrac{G_2{}^*}{h_2}\left(u_3-u_1-h\dfrac{\partial w}{\partial x}\right)=m_3\dfrac{\partial^2 u_3}{\partial t^2} \\ K_3\left(\dfrac{\partial^2 v_3}{\partial y^2}+\dfrac{1+\mu_3}{2}\dfrac{\partial^2 u_3}{\partial x\partial y}+\dfrac{1-\mu_3}{2}\dfrac{\partial^2 v_3}{\partial x^2}\right)-\dfrac{G_2{}^*}{h_2}\left(v_3-v_1-h\dfrac{\partial w}{\partial y}\right)=m_3\dfrac{\partial^2 v_3}{\partial t^2} \\ -\dfrac{hG_2{}^*}{h_2}\left(\dfrac{\partial u_3}{\partial x}-\dfrac{\partial u_1}{\partial x}+\dfrac{\partial v_3}{\partial y}-\dfrac{\partial v_1}{\partial y}\right)-(D_1+D_3)\left(\dfrac{\partial^4 w}{\partial x^4}+\dfrac{\partial^4 w}{\partial y^4}+2\dfrac{\partial^4 w}{\partial x^2\partial y^2}\right)+\dfrac{h^2 G_2{}^*}{h_2}\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\right)=m\dfrac{\partial^2 w}{\partial t^2} \end{cases} \tag{9}$$

式中：

$$m_1=\rho_1 h_1,\ m_2=\rho_2 h_2,\ m_3=\rho_3 h_3,\ m=m_1+m_2+m_3,\ K_1=\frac{E_1 h_1}{1-\mu_1{}^2}$$

$$K_3=\frac{E_3 h_3}{1-\mu_3{}^2},\ D_1=\frac{E_1 h_1^3}{12(1-\mu_1^2)},\ D_3=\frac{E_3 h_3^3}{12(1-\mu_3^2)},\ h=\frac{(h_1+h_3)}{2}+h_2$$

2 解析法求解

对于四边简支板，有边界条件：$x=0,\ a$：$v_1=v_3=w=0$，$y=0,b$：$u_1=u_3=w=0$，则方程(9)的解可写成式(10)。式中 m、n 分别为 x、y 方向的半波数。$\omega*$ 是复频率，$\omega*=\omega^2(1+\eta i)$。$\omega$ 为固有振动圆频率，η 为结构损耗因子。

$$\begin{cases} u_1(x,y,t)=U_1\cos\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}e^{i\omega^* t} \\ v_1(x,y,t)=V_1\sin\dfrac{m\pi x}{a}\cos\dfrac{n\pi y}{b}e^{i\omega^* t} \\ u_3(x,y,t)=U_3\cos\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}e^{i\omega^* t} \\ v_3(x,y,t)=V_3\sin\dfrac{m\pi x}{a}\cos\dfrac{n\pi y}{b}e^{i\omega^* t} \\ w(x,y,t)=W\sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}e^{i\omega^* t} \end{cases} \quad (10)$$

将式(10)代入方程(9)可得：

$$\begin{cases} \left[m_1\omega^{*2}-K_1\left(\dfrac{m\pi}{a}\right)^2-K_1\dfrac{1-\mu_1}{2}\left(\dfrac{n\pi}{a}\right)^2-\dfrac{G_2^*}{h_2}\right]U_1-\left[K_1\dfrac{1+\mu_1}{2}\dfrac{m\pi}{a}\dfrac{n\pi}{b}\right]V_1+\dfrac{G_2{}^*}{h_2}U_3-\left[\dfrac{hG_2{}^*}{h_2}\dfrac{m\pi}{a}\right]W=0 \\ \left[-K_1\dfrac{1+\mu_1}{2}\dfrac{m\pi}{a}\dfrac{n\pi}{b}\right]U_1+\left[m_1\omega^{*2}-K_1\left(\dfrac{n\pi}{a}\right)^2-K_1\dfrac{1-\mu_1}{2}\left(\dfrac{m\pi}{a}\right)^2-\dfrac{G_2{}^*}{h_2}\right]V_1+\dfrac{G_2{}^*}{h_2}V_3-\left[\dfrac{hG_2{}^*}{h_2}\dfrac{n\pi}{b}\right]W=0 \\ \dfrac{G_2{}^*}{h_2}U_1+\left[m_3\omega^{*2}-K_3\left(\dfrac{m\pi}{a}\right)^2-K_3\dfrac{1-\mu_3}{2}\left(\dfrac{n\pi}{a}\right)^2-\dfrac{G_2{}^*}{h_2}\right]U_3-\left[K_3\dfrac{1+\mu_3}{2}\dfrac{m\pi}{a}\dfrac{n\pi}{b}\right]V_3+\left[\dfrac{hG_2{}^*}{h_2}\dfrac{m\pi}{a}\right]W=0 \\ \dfrac{G_2{}^*}{h_2}V_1-\left[K_3\dfrac{1+\mu_3}{2}\dfrac{m\pi}{a}\dfrac{n\pi}{b}\right]U_3+\left[m_3\omega^{*2}-K_3\left(\dfrac{n\pi}{a}\right)^2-K_3\dfrac{1-\mu_3}{2}\left(\dfrac{m\pi}{a}\right)^2-\dfrac{G_2{}^*}{h_2}\right]V_3+\left[\dfrac{hG_2{}^*}{h_2}\dfrac{n\pi}{b}\right]W=0 \\ \left[\dfrac{hG_2{}^*}{h_2}\dfrac{m\pi}{a}\right]U_1+\left[\dfrac{hG_2{}^*}{h_2}\dfrac{n\pi}{b}\right]V_1-\left[\dfrac{hG_2{}^*}{h_2}\dfrac{m\pi}{a}\right]U_3-\left[\dfrac{hG_2{}^*}{h_2}\dfrac{n\pi}{b}\right]V_3+\left[-m\omega^{*2}+(D_1+D_3)\left(\left(\dfrac{m\pi}{a}\right)^4+\left(\dfrac{n\pi}{a}\right)^4+\right.\right. \\ \left.\left.2\left(\dfrac{m\pi}{a}\right)^2\left(\dfrac{n\pi}{a}\right)^2\right)+\dfrac{h^2G_2{}^*}{h_2}\left(\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right)\right]W=0 \end{cases} \quad (11)$$

要使方程组(11)有解需系数行列式为零，解方程可得 ω^*，可求得约束阻尼板的固有频率和结构损耗因子。

3 传递矩阵法求解

两对边为简支时，可采用传递矩阵法求解，假设为 y 方向，则方程(11)的解可写成式(12)，其中 $U_1(x)$，$V_1(x)$，$U_3(x)$，$V_3(x)$，$W(x)$分别为 x 的函数，与边界条件有关。

$$\begin{cases} u_1(x,y,t)=U_1(x)\sin\dfrac{n\pi y}{b}e^{i\omega^* t} \\ v_1(x,y,t)=V_1(x)\cos\dfrac{n\pi y}{b}e^{i\omega^* t} \\ u_3(x,y,t)=U_3(x)\sin\dfrac{n\pi y}{b}e^{i\omega^* t} \\ v_3(x,y,t)=V_3(x)\cos\dfrac{n\pi y}{b}e^{i\omega^* t} \\ w(x,y,t)=W(x)\sin\dfrac{n\pi y}{b}e^{i\omega^* t} \end{cases} \quad (12)$$

将式(12)代入方程(9)并推导可得：

$$\begin{cases}\dfrac{\partial^2 U_1}{\partial x^2}=\dfrac{1-\mu_1}{2}\left(\dfrac{n\pi}{b}\right)^2 U_1+\dfrac{1+\mu_1}{2}\dfrac{n\pi}{b}\dfrac{\partial V_1}{\partial x}-\dfrac{G_2^{\ *}}{K_1 h_2}\left(U_3-U_1-h\dfrac{\partial W}{\partial x}\right)-\dfrac{m_1 U_1\omega^{*2}}{K_1}\\ \dfrac{\partial^2 V_1}{\partial x^2}=\dfrac{2}{1-\mu_1}\left(-\dfrac{G_2^{\ *}}{K_1 h_2}\left(V_3-V_1-h\dfrac{n\pi}{b}W\right)-\dfrac{m_1}{K_1}V_1\omega^{*2}+\left(\dfrac{n\pi}{b}\right)^2 V_1-\dfrac{1+\mu_1}{2}\dfrac{n\pi}{b}\dfrac{\partial U_1}{\partial x}\right)\\ \dfrac{\partial^2 U_3}{\partial x^2}=\dfrac{1+\mu_3}{2}\dfrac{n\pi}{b}\dfrac{\partial V_3}{\partial x}+\dfrac{1-\mu_3}{2}\left(\dfrac{n\pi}{b}\right)^2 U_3+\dfrac{G_2^{\ *}}{K_3 h_2}\left(U_3-U_1-h\dfrac{\partial W}{\partial x}\right)-\dfrac{m_3}{K_3}U_3\omega^{*2}\\ \dfrac{\partial^2 V_3}{\partial x^2}=\dfrac{2}{1-\mu_3}\left(\dfrac{G_2^{\ *}}{K_3 h_2}\left(V_3-V_1-h\dfrac{n\pi}{b}W\right)-\dfrac{m_3}{K_3}V_3\omega^{*2}+\left(\dfrac{n\pi}{b}\right)^2 V_3-\dfrac{1+\mu_3}{2}\dfrac{n\pi}{b}\dfrac{\partial U_3}{\partial x}\right)\\ \dfrac{\partial^4 W}{\partial x^4}=\dfrac{-1}{D_1+D_3}\left(\begin{aligned}&-mW\omega^{*2}+\dfrac{hG_2^{\ *}}{h_2}\left(\dfrac{\partial U_3}{\partial x}-\dfrac{\partial U_1}{\partial x}-\left(\dfrac{n\pi}{b}\right)V_{3n}+\left(\dfrac{n\pi}{b}\right)V_{1n}\right)+\\&\dfrac{h^2G_2^{\ *}}{h_2}\left(\dfrac{\partial^2 W}{\partial x^2}-\left(\dfrac{n\pi}{b}\right)^2 W\right)\end{aligned}\right)-\left(\dfrac{n\pi}{b}\right)^4 W+2\left(\dfrac{n\pi}{b}\right)^2\dfrac{\partial^2 W}{\partial x^2}\end{cases}\tag{13}$$

边界条件为：$x=0$，a 时，
$$\begin{cases}U_1=0\text{ 或 }N_{x1}=0\quad V_1=0\text{ 或 }N_{xy1}=0\quad U_3=0\text{ 或 }N_{x3}=0\quad V_3=0\text{ 或 }N_{xy3}=0\\ \dfrac{\partial W}{\partial x}=0\text{ 或 }M_x=M_{x1}+M_{x3}=0\\ W=0\text{ 或 }Q_x=Q_{x1}+Q_{x3}=0\end{cases}\tag{14}$$

传统传递矩阵法选择边界条件向量 $\xi(x)=\left[U_1,N_{x1},V_1,N_{xy1},U_3,N_{x3},V_3,N_{xy3},W,Q_x,\dfrac{\partial W}{\partial x},M_x\right]^T$ 为状态向量，直接求解状态向量的一阶倒数$\dfrac{\partial\xi(x)}{\partial x}$将非常困难，且向量个数越多时求解越难。本文对传递矩阵法进行改进，分别定义状态向量选择状态向量 $\xi(x)$和边界条件向量 $\zeta(x)$如下：

$$\xi(x)=\left[U_1,\frac{\partial U_1}{\partial x},V_1,\frac{\partial V_1}{\partial x},U_3,\frac{\partial U_3}{\partial x},V_3,\frac{\partial V_3}{\partial x},W,\frac{\partial W}{\partial x},\frac{\partial^2 W}{\partial x^2},\frac{\partial^3 W}{\partial x^3}\right]^T\quad \zeta(x)=\left[U_1 N_{x1} V_1 N_{xy1} U_3 N_{x3} V_3 N_{xy3} W Q_x \frac{\partial W}{\partial x} M_x\right]^T$$

则有：$\dfrac{\partial\xi(x)}{\partial x}=F\xi(x)$，$\xi(x)=e^{Fx}\xi(0)$。F 为一个 12 阶方阵，可由方程(13)得到。$\zeta(x)=A\xi(x)$，A 为 12 阶方阵，可由式(14)得到

$$\zeta(a)=A\xi(a)=Ae^{Fa}\xi(0)=Ae^{Fa}A^{-1}A\xi(0)=TA\xi(0)=T\zeta(0)$$

式中 T 即为传递矩阵。通过在边界条件向量和状态向量之间引入一个关联矩阵 A，使得状态向量的一阶导数求解非常方便，而 A 也可根据边界条件直接得到，这就大大简化了传递矩阵 T 的求解，尤其是在多夹层结构中，向量元素个数更多时，更加明显。

两端简支边界条件下，有：

$$A\zeta(a)=\left[U_1\ 0\ U_3\ 0\ \frac{\partial W}{\partial x}\ 0\ 0\ N_{xy1}\ 0\ N_{xy3}\ 0\ Q_x\right]^T A\zeta(0)=\left[U_1\ 0\ U_3\ 0\ \frac{\partial W}{\partial x}\ 0\ 0\ N_{xy1}\ 0\ N_{xy3}\ 0\ Q_x\right]^T \text{ 即有：}$$

$$\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}T_{2,1}&T_{2,3}&T_{2,5}&T_{2,8}&T_{2,10}&T_{2,12}\\T_{4,1}&T_{4,3}&T_{4,5}&T_{4,8}&T_{4,10}&T_{4,12}\\T_{6,1}&T_{6,3}&T_{6,5}&T_{6,8}&T_{6,10}&T_{6,12}\\T_{7,1}&T_{7,3}&T_{7,5}&T_{7,8}&T_{7,10}&T_{7,12}\\T_{9,1}&T_{9,3}&T_{9,5}&T_{9,8}&T_{9,10}&T_{9,11}\\T_{11,1}&T_{11,3}&T_{11,5}&T_{11,8}&T_{11,10}&T_{11,11}\end{bmatrix}\begin{bmatrix}U_1\\U_3\\\frac{\partial W}{\partial x}\\N_{xy1}\\N_{xy3}\\Q_x\end{bmatrix}=T'\begin{bmatrix}U_1\\U_3\\\frac{\partial W}{\partial x}\\N_{xy1}\\N_{xy3}\\Q_x\end{bmatrix}\tag{15}$$

要使式(15)有非零解，则有 T' 行列式为零，即可解得频率和结构损耗因子。不同的边界条件 T' 不同。

4 算例

为了验证计算方法和程序的正确性，选 C D Johnson 给出的一个实例。一对称的各向同性板，采用约束阻尼结构，其几何参数和物理参数为：$a=0.3048m$，$b=0.3480m$，$h_1=h_3=0.762mm$，$h_2=0.254mm$，$E_1=E_3=6.89\times10^{10}pa$，$\mu_1=\mu_3=0.3$，$G_2=0.896\times(1+0.5)\times10^6 pa$，$\rho_1=\rho_3=2740kg/m^3$，$\rho_2=999kg/m^3$。

表 1 列出原文中理论解、有限元解、本文解析解、传递矩阵法解。

5 结论

1)从表 1 中,可看出,传递矩阵法计算精度很高。

2)传递矩阵法对边界条件的适应性较好,而解析法只能计算四边简支板。

3)改进后的传递矩阵法通过引入一个关联矩阵,使状态向量一阶倒数的求解大大简化,也进一步简化了传递矩阵的求解。

4)本文对三层阻尼结构的求解方法,可为多夹层阻尼结构的求解及振动分析奠定基础。

表 1 约束阻尼板动力学特性不同解法计算结果

模态	原文理论解		原文有限元解		本文解析解		传递矩阵解	
M,n	f/Hz	η	f/Hz	η	f/Hz	η	f/Hz	η
1,1	60.3	0.190	57.4	0.176	60.2	0.190	60.2	0.190
1,2	115.4	0.203	113.2	0.188	115.2	0.203	115.2	0.204
2,1	130.6	0.199	129.3	0.188	130.4	0.199	130.4	0.199
2,2	178.7	0.181	179.3	0.153	178.5	0.181	178.5	0.181
1,3	195.7	0.174	196.0	0.153	195.4	0.174	195.4	0.174

参考文献:

[1] 戴德沛. 阻尼技术的工程应用[M]. 北京:清华大学出版社,1991.
Dai D P. Engineering application of damping technology [M]. Beijing: Tsinghua University Publishing House, 1991.

[2] He J F, Ma S B A. Analysis of flexural vibration of viscoelastically damped sandwich plates [J]. Journal of Sound and Vibration, 1988, 126: 37-47.

[3] Cupia P, Niziol J. Vibration and damping analysis of a three-layered composite plate with a viscoelastic mid-layer [J]. Journal of Sound and Vibration, 1995, 183(1): 99-114.

[4] Wang H J, Chen L W. Vibration and damping analysis of a three-layered composite annular plate with a viscoelastic mid-layer [J]. Composite Structures, 2002, 58: 563-570.

[5] 钱振东,陈国平,朱德懋. 约束阻尼层板的振动分析[J]. 南京航空航天大学学报,1997,25(7):517-522.
Qian Z D, Chen G P, Zhu D M. Vibration analysis of plate attached to constrained damping layer [J]. Journal of Nanjing University of Aeronautics & Astronautics. 1997, 25 (7): 517-522.

[6] 李恩奇,唐国金,雷勇军,等. 约束层阻尼板动力学问题的传递函数解[J]. 国防科技大学学报. 2008,30(1):5-9.
Li E Q, Tang G J, Lei Y J, et al. Dynamic analysis of constrained layer damping plate by the transfer function method[J]. Journal of National University of Defense Technology, 2008, 30(1): 5-9.

[7] 唐国金,李恩奇,李道奎,等. 约束层阻尼板动力学问题的半解析解[J]. 固体力学学报. 2008,29(2):149-156.
Tang G J, Li E Q, Li D K, et al. Semi-analytical dynamics solution of constrained layer damping plate [J]. Chinese Journal of Solid Mechanics, 2008, 29 (2): 149-156.

[8] 万浩川,李以农,郑玲. 分析结构振动的改进传递矩阵法[J]. 振动与冲击,2013,32(9):173-177.
Wan H C, Li Y O, Zheng L. Improved transfer matrix method in structural vibration analysis [J]. Journal of Vibration and Shock, 2013, 32(9): 173-177.

The Analytical Method and Improved Transfer Matrix Method of Constrained Damping Plate

WAN Haochuan[1], LI Yinong[2], ZHENG ling[2]

(1. State Key Laboratory of Mechanical Transmission, Chongqing University, Chongqing 400044;
2. Leshan Vocational & Technical College, Leshan Sichuan 614000, China)

Abstract: Based on the Kirchhoff hypothesis and Kerwin hypothesis, the vibration equations of three-layered constrained damping plate are established. The equations are solved by analytical method and improved transfer matrix method. The results show that the methods are accurate. The transfer matrix method can be used in more boundary conditions. By introducing an association matrix, the first order derivative of the state vector and the transfer matrix can be obtained more easily, and the advantage is more obvious in the multi-layered structure. The method in this paper can be used in the compute and analysis of multi-layered constrained damping structure.

Key words: constrained damping plate; analytical method; transfer matrix method; association matrix

(责任编辑 李若溪)