DOI:10.11721/cqnuj20150114

A Preconditioned Matrix Splitting Method for Solving Saddle Point Problem with Indefinite (1,1) Block*

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Abstract: we propose a preconditioned of the Gill-Murray forced definite method, which forces a symmetric indefinite matrix to a positive definite matrix. Then the splitting is used to construct an iterative method, that is the coefficient matrix is multiplied by a preconditioned matrix $\boldsymbol{P}$, then the coefficient matrix is split and get an iterative matrix, The Iteration method is used for solving the saddle point problems which (1,1) block is indefinite in the coefficient matrix. Under suitable conditions, we prove the convergence of the new preconditioned iterative method. Finally, this paper shows that the resulting new preconditioned method leads to fast convergence.

Key words: symmetric indefinite; Gill-Murray forced definite method; preconditioned
中图分类号:O246 **文献标志码**:A **文章编号**:1672-6693(2015)01-0072-04

1 Introduction

In this paper, we consider the iterative solution for two by two block linear system of the form

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1}$$

where $\boldsymbol{A}\in\mathbf{R}^{n\times n}$, is symmetric indefinite, $\boldsymbol{B}\in\mathbf{R}^{m\times n}(n\geqslant m)$ has row full rank, that is rank $(\boldsymbol{B})=m$, vectors x, $f\in\mathbf{R}^{n}$, y, $g\in\mathbf{R}^{m}$. Under these assumptions, the above (1) has a unique solution. This linear system is called a saddle point problem.

Saddle point problems arises in many different applications scientific computing fields and engineering applications fields such as constrained optimization, computational fluid dynamics, mixed finite element methods for solving elliptic partial differential equations and Stokes problems, constrained least-squares problems, structure analysis and so forth[1-8]. When $\boldsymbol{A}\in\mathbf{R}^{n\times n}$ is a symmetric positive definite matrix in the linear system (1), this problem many papers have discussed[9-10].

In this paper, a preconditioned of the Gill-Murray forced definite method is considered for solving the saddle point problem with symmetric indefinite $\boldsymbol{A}\in\mathbf{R}^{n\times n}$, the remainder of the paper is organized as follows. In section 2, we propose preconditioned iterative method. In section 3, we discuss the conditions for guaranteeing its convergence. In section 4, is given to show the new method is feasible and efficient.

2 Preconditioned iterative method

In this section, we present a preconditioned iterative method for the saddle point problem (1) when $\boldsymbol{A}\in$

* **Received**:10-06-2013 **Accepted**:08-03-2014 **网络出版时间**:2015-1-7 16:04
Foundation: Science Foundation of Qujing Normal University(No. 2011QNZC2); Science Research Foundation of Yunnan Province Education Department(No. 2012Y418)
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收稿日期:2013-10-06 **修回日期**:2014-08-03 **网络出版时间**:2015-1-7 16:04
资助项目:曲靖师范学院科学研究基金(No. 2011QNZC2);云南省教育厅科学研究基金(No. 2012Y418)
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$\mathbf{R}^{n\times n}$, is symmetric indefinite.

For the sake of simplicity, we rewrite the saddle point system (1) as

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}} \\ -\boldsymbol{B} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} f \\ -g \end{bmatrix}. \tag{2}$$

The Gill-Murray forced definite method has been proposed[6], which construct the following matrix splitting

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}} \\ -\boldsymbol{B} & \boldsymbol{0} \end{bmatrix}=\begin{bmatrix} \boldsymbol{LDL}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{M}-\boldsymbol{B} & \boldsymbol{Q} \end{bmatrix}-\begin{bmatrix} \boldsymbol{E} & -\boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{M} & \boldsymbol{Q} \end{bmatrix}. \tag{3}$$

Multiplying both sides of the linear system with left preconditioned matrix $\hat{\boldsymbol{P}}=\begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$, then we have

$$\begin{bmatrix} \boldsymbol{PA} & \boldsymbol{PB}^{\mathrm{T}} \\ -\boldsymbol{B} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \boldsymbol{P}f \\ -g \end{bmatrix}. \tag{4}$$

Let $\boldsymbol{A}\in\mathbf{R}^{n\times n}$, be symmetric indefinite. Then from the Gill-Murray forced definite decomposition[1], there exists a decomposition which forced positive definite, i. e.

$$\boldsymbol{A}=\boldsymbol{LDL}^{\mathrm{T}}-\boldsymbol{E} \tag{5}$$

Where $\boldsymbol{L}$ is identity lower triangular, $\boldsymbol{D}$ is positive diagonal, and $\boldsymbol{LDL}^{\mathrm{T}}$ is symmetric positive definite, $\boldsymbol{E}$ is diagonal.

From(5), let $\boldsymbol{P}=(\boldsymbol{LDL}^{\mathrm{T}})^{-1}$, we use the decomposition (4) to construct the following matrix splitting

$$\begin{bmatrix} \boldsymbol{PA} & \boldsymbol{PB}^{\mathrm{T}} \\ -\boldsymbol{B} & \boldsymbol{0} \end{bmatrix}=\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{M}-\boldsymbol{B} & \boldsymbol{Q} \end{bmatrix}-\begin{bmatrix} (\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E} & -(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{M} & \boldsymbol{Q} \end{bmatrix}. \tag{6}$$

Where $\boldsymbol{Q}$ is symmetric and nonsingular, $\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{M}-\boldsymbol{B} & \boldsymbol{Q} \end{bmatrix}$ is nonsingular.

By using the splitting(7), we propose the following iterative method for solving (2)

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{M}-\boldsymbol{B} & \boldsymbol{Q} \end{bmatrix}\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}=\begin{bmatrix} (\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E} & -(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{M} & \boldsymbol{Q} \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}+\begin{bmatrix} (\boldsymbol{LDL}^{\mathrm{T}})^{-1}f \\ -g \end{bmatrix} \tag{7}$$

This iterative method can be written as following algorithm.

Algorithm

Step Ⅰ Using the Gill-Murray forced definite algorithm[1], produce the decomposition $\boldsymbol{A}=\boldsymbol{LDL}^{\mathrm{T}}-\boldsymbol{E}$;

Step Ⅱ Choose matrices $\boldsymbol{M}\in\mathbf{R}^{m\times n}$, $\boldsymbol{P}\in\mathbf{R}^{n\times n}$, and $\boldsymbol{M}$, construct matrix splitting (6);

Step Ⅲ Given initial vectors $x_0\in\mathbf{R}^n$ and $y_0\in\mathbf{R}^m$, for $k=0,1,2,\cdots,\{(x_k^{\mathrm{T}}, y_k^{\mathrm{T}})^{\mathrm{T}}\}$ is produced by the following scheme

$$\begin{cases} x_{n+1}=(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E}x_n-(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}y_n+(\boldsymbol{LDL}^{\mathrm{T}})^{-1}f \\ y_{n+1}=y_n-\boldsymbol{Q}^{-1}(\boldsymbol{M}-\boldsymbol{B})x_{n+1}+\boldsymbol{Q}^{-1}\boldsymbol{M}x_n-\boldsymbol{Q}^{-1}g \end{cases}; \tag{8}$$

The iteration matrix is $\boldsymbol{G}=\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{M}-\boldsymbol{B} & \boldsymbol{Q} \end{bmatrix}^{-1}\begin{bmatrix} (\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E} & -(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{M} & \boldsymbol{Q} \end{bmatrix}$.

3 Convergence analysis

In the following, we discuss the convergence property of the iterative method (8). Let $\rho(\boldsymbol{G})$ denote the spectral radius of the iterative matrix $\boldsymbol{G}$. Then this method converges if and only if $\rho(\boldsymbol{G})<1$. Let λ be an eigenvalue of $\boldsymbol{G}$ and $(u_k^{\mathrm{T}}, v_k^{\mathrm{T}})^{\mathrm{T}}$ be the corresponding eigenvector, that is

$$\begin{bmatrix} (\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E} & -(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{M} & \boldsymbol{Q} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{M}-\boldsymbol{B} & \boldsymbol{Q} \end{bmatrix}\lambda\begin{bmatrix} u \\ v \end{bmatrix}, \tag{9}$$

or equivalently

$$\begin{cases} (\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E}u-(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}v=lu \\ [\boldsymbol{M}-l(\boldsymbol{M}-\boldsymbol{B})]u=(l-1)\boldsymbol{Q}v \end{cases}. \tag{10}$$

Lemma[3] Both roots of the quadratic equation $x^2+bx+c=0$ have modulus less than one if and only if $|c|<1$ and $|b|<1+c$.

Now, we present the convergence theorem.

Theorem Assume that $\boldsymbol{A}\in\mathbf{R}^{n\times n}$ is symmetric indefinite and $\boldsymbol{A}=\boldsymbol{LDL}^{\mathrm{T}}-\boldsymbol{E}$, where $\boldsymbol{L}$ is identity lower triangular, $\boldsymbol{D}$ is positive diagonal, $\boldsymbol{E}$ is diagonal, $\boldsymbol{B}\in\mathbf{R}^{m\times n}(n\geqslant m)$ has full row rank, $\boldsymbol{Q}$ are nonsingular and symmetric. Let λ be an eigenvalue of $\boldsymbol{G}$ and $(u^{\mathrm{T}}, v^{\mathrm{T}})^{\mathrm{T}}$ be the corresponding eigenvector, then the iterative method (8) is convergent if and only if

$$\begin{cases}|\alpha_{\max}+\beta_{\max}|<1\\0<\gamma<2(1+\alpha_{\max}+\beta_{\max})\end{cases}, \tag{11}$$

where $\alpha=\frac{u\,(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E}u^{\mathrm{T}}}{u^{\mathrm{T}}u}$, $\beta=\frac{u^{\mathrm{T}}\,(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{M}u}{u^{\mathrm{T}}u}$, $\gamma=\frac{u^{\mathrm{T}}\,(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{B}u}{u^{\mathrm{T}}u}$ and $\alpha_{\max}$, $\beta_{\max}$ are the largest eigenvalues of $(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E}$ and $(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{M}$.

Proof From (10), we have $(\lambda-1)v=(1-\lambda)\boldsymbol{Q}^{-1}\boldsymbol{M}u+\lambda\boldsymbol{Q}^{-1}\boldsymbol{B}u$, and

$$(\lambda-1)(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E}u-(1-\lambda)(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{M}u-\lambda\,(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{B}u=\lambda(\lambda-1)u. \tag{12}$$

Multiplying both sides of this equation from left with $\frac{u^{\mathrm{T}}}{u^{\mathrm{T}}u}$, we have

$$(\lambda-1)\frac{u^{\mathrm{T}}\,(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{E}u}{u^{\mathrm{T}}u}-(1-\lambda)\frac{u^{\mathrm{T}}\,(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{M}u}{u^{\mathrm{T}}u}-\lambda\,\frac{u^{\mathrm{T}}\,(\boldsymbol{LDL}^{\mathrm{T}})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{B}u}{u^{\mathrm{T}}u}=\lambda(\lambda-1)\frac{u^{\mathrm{T}}u}{u^{\mathrm{T}}u}. \tag{13}$$

That is $\lambda^2-(1+\alpha+\beta-\gamma)\lambda+\alpha+\beta=0$.

From the above lemma, $|\lambda|<1$ if and only if $\begin{cases}|\alpha+\beta|<1\\|1+\alpha+\beta-\gamma|<1+\alpha+\beta\end{cases}$.

Hence the iteration method (8) converges if and only if $\begin{cases}|\alpha_{\max}+\beta_{\max}|<1\\0<\gamma<2(1+\alpha_{\max}+\beta_{\max})\end{cases}$.

The proof of thetheorem is completed.

4 Numerical example

In this section, we give a numerical example to show that the iteration method (8) is feasible and effective.

Example Let $\boldsymbol{A}=(a_{ij})$, $\boldsymbol{B}=(b_{ij})$, where

$$(a_{ij})_{n\times n}=\begin{cases}i-1, |i-j|=1\\-i^3-1,\ i=j\\0,\ \text{others}\end{cases},\quad (b_{ij})_{n\times n}=\begin{cases}i,\ i=j\\i-1,\ |i-j|=1\\0,\ \text{others}\end{cases}.$$

Then the conditions of the above theorem are satisfied. We use a zero initial guess and stop the iteration as soon as the relative residual is less than 10^{-5}, the corresponding numerical results are listed in Tab. 1.

Tab. 1 The spectral radii and the seconds needed for convergence

N	$\rho(\boldsymbol{G})$	CPU	IT
50	0.085 6	0.063 0	310
100	0.085 6	0.282 0	335
150	0.085 6	0.773 0	452

In Tab. 1, we list the spectral radius ρ of the iterative matrix $\boldsymbol{G}$, IT denotes the iteration steps and the seconds needed for convergence of the iteration method (8) for different values of N. Tab. 1 shows that the iterative method (8) is convergent, and the new method is effective.

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解(1,1)块不定鞍点问题的预条件分裂方法

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摘要: 提出了一类吉尔-默里强迫正定的预条件方法,该方法是使一个对称不定矩阵强迫分裂出一个正定矩阵,然后用该分裂方法构造一个迭代方法用于求解在系数矩阵中(1,1)块为不定的鞍点问题,在合适的条件下,证明了新的预条件迭代法的收敛性,最后,数值算例表明新预条件方法具有的收敛性。

关键词: 对称不定;吉尔-默里强迫正定方法;预条件

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