2015 年 3 月　　重庆师范大学学报(自然科学版)　　Mar. 2015
第 32 卷 第 2 期　　Journal of Chongqing Normal University (Natural Science)　　Vol. 32 No. 2

DOI:10.11721/cqnuj20150219

The Limiting Distribution of Finite Mixture General Error Eistribution*

HUANG Jianwen, LIU Yanmin, LUO Yuanfeng
(School of Mathematics and Computational Science, Zunyi Normal College, Zunyi Guizhou 563002, China)

Abstract: The general error distribution being a generalization of normal distribution is one of the most widely applied distributions in statistics. Finite mixture distribution as a model has been widely studied and applied in throughout the development process of modern statistics. In this paper, based on an inequality of the general error distribution, the asymptotic tail behavior of finite mixture general error distribution is studied and the asymptotic behavior of the tail of this distribution is derived while Mills' ratio is obtained. Then, an application of the asymptotic property of the tail of the distribution to consider the limiting distribution of the extreme of independent identically distributed (i. i. d.) random variable sequence with finite mixed general error distribution and how to choose the suitable normalized constants such that the distribution of maxima converges to double exponential distribution.

Key words: general error distributions; extreme value distribution; mixture distribution; tail behavior
中图分类号:O211.4　　**文献标志码**:A　　**文章编号**:1672-6693(2015)02-0095-04

1 Introduction

General error distribution[1] (for short GED) has been widely used in modeling volatility of high-frequency time series with heavy tail since Nelson used it in his exponential general autoregressive conditionally heteroscedastic model. The probability density function (pdf) of the standardized GED is given by:

$$f(x)=\frac{v\exp\{-(1/2)|x/\lambda|^{v}\}}{\lambda 2^{1+1/v}\Gamma(1/v)},$$

for $v>0$ and $x\in\mathbf{R}$, where $\lambda=[2^{-2/v}\Gamma(1/v)/\Gamma(3/v)]^{1/2}$ and $\Gamma(\cdot)$ denotes the Gamma function (cf. [1], p. 352). Nelson also pointed that the GED reduces to the standard normal distribution when $v=2$, and v represents the tail thickness parameter, i. e., for $0<v<2$ the tail of the GED is thicker than that of the normal distribution, and the GED has a thinner tail for $v>2$.

A finite mixture distribution is defined as follows. Let $X_1, X_2, \cdots, X_s$ be independent random variables with cumulative distribution functions (cdf) $F_i(x), i=1,2,\cdots,s$. Define a new random variable Z by

$$Z=\begin{cases} X_1, & \text{with probability } p_1 \\ X_2, & \text{with probability } p_2 \\ \cdots & \cdots \\ X_s, & \text{with probability } p_s \end{cases},$$

* **Received**:11-21-2013　**Accepted**:12-25-2014　**网络出版时间**:2015-01-22 11:56
Foundation:Project supported by NSFC (No. 71461027); the Science and Technology Fund Project of GZ(No. LKZS[2014]22; No. LKZS[2014]29)
The first author biography: Huang Jianwen, male, lecturer, research area: extreme value theory and limit theorem, E-mail: hjw1303987297@126.com
收稿日期:2013-11-21　**修回日期**:2014-12-25　**网络出版时间**:2015-01-22 11:56
资助项目:国家自然科学基金(No. 71461027);贵州省科学技术基金项目(No. LKZS[2014]22;No. LKZS[2014]29)
作者简介:黄建文,男,讲师,研究方向为极值理论和极值定理,E-mail:hjw1303987297@126.com
网络出版地址:http://www.cnki.net/kcms/detail/50.1165.N.20150122.1156.011.html

then Z is said to have a mixed distribution with s components and weights $p_i \geqslant 0$ for $1 \leqslant i \leqslant s$ satisfying $\sum_{i=1}^{s} p_i = 1$. It is easy to check that the cdf of Z is given by

$$F(x) = p_1 F_1(x) + p_2 F_2(x) + \cdots + p_s F_s(x). \tag{1}$$

An particular case of (1) arises when $F_k(x)(k=1,2,\cdots,s)$ is an general error distribution with parameter $v_k > 0$. Then Z is said to have the mixed general error distribution (for short MGED) with $s$components.

Mladenovic'[2] investigated the limiting distributions of maximum of i. i. d. random sequence with some mixed distributions of two components. Cheng and Peng[3] extended the above results to the general finite mixed distributions. Liu and Peng [4] discussed the tail behavior of finite mixed distribution from the short-tail symmetric distribution and the limiting distribution of maximum.

The aim of this paper is to establish the tail property of mixture general error distribution and the limiting distribution of maxima of independent identically random sequence with common mixed distribution defined by (1).

An inequality on the tail behavior of the GED and an asymptotic Mills-type ratio have been studied by Peng et al[5]. Denote by $\hat{F}(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ the cdf of the GED. The symmetric property of $f(x)$ implies $\hat{F}(-x) = 1 - \hat{F}(x)$. Let $v > 1$. For all $x > 0$, we have

$$\frac{2\lambda^v}{v} x^{1-v} \left(1 + \frac{2(v-1)\lambda^v}{v} x^{-v}\right)^{-1} < \frac{\hat{F}(-x)}{f(x)} < \frac{2\lambda^v}{v} x^{1-v}, \tag{2}$$

where $\lambda = [2^{-2/v}\Gamma(1/v)/\Gamma(3/v)]^{1/2}$.

2 Tail behavior of the finite mixture GED

Theorem 1 $F(x)$ is MGED which determined by (1). Then

$$1 - F(t) = \sum_{i=1}^{s} p_i (1 - F_i(t)) \sim w t^{1-v} \exp\{-(1/2)(t/\lambda)^v\},$$

where $v = \min\{v_i, i=1,2,\cdots,s\}$, λ is the corresponding $\lambda_i (i=1,2,\cdots,s)$ when $v = \min\{v_i, i=1,2,\cdots,s\}$, $w = \sum_{i:v=v_i} p_i \lambda_i^{v_i-1} / (2^{1/v_i}\Gamma(1/v_i))$.

Proof Since

$$1 - F(t) = 1 - \sum_{i=1}^{s} p_i F_i(t) = \sum_{i=1}^{s} p_i (1 - F_i(t)), \tag{3}$$

using inequality (2), we have

$$\frac{2\lambda_i^{v_i}}{v_i} t^{1-v_i} \left(1 + \frac{2(v_i-1)\lambda_i^{v_i}}{v_i} t^{-v_i}\right)^{-1} < \frac{1-F_i(t)}{f_i(t)} < \frac{2\lambda_i^{v_i}}{v_i} t^{1-v_i}, \tag{4}$$

where $f_i(x)$ is the corresponding density function of the distribution function $F_i(x)$, $i=1,2,\cdots,s$.

Combining with (3) and (4), we have

$$A(t) = \sum_{i=1}^{s} p_i \frac{2\lambda_i^{v_i}}{v_i} t^{1-v_i} f_i(t) \left(1 + \frac{2(v_i-1)\lambda_i^{v_i}}{v_i} t^{-v_i}\right)^{-1} < \sum_{i=1}^{s} p_i (1 - F_i(t)) <$$
$$\sum_{i=1}^{s} p_i \frac{2\lambda_i^{v_i}}{v_i} t^{1-v_i} f_i(t) = B(t).$$

Then, as $t \to \infty$,

$$A(t) = \sum_{i=1}^{s} p_i \frac{\lambda_i^{v_i-1}}{2^{1/v_i}} t^{1-v_i} \Gamma(1/v_i) \exp\left\{-\frac{1}{2}\left(\frac{t}{\lambda_i}\right)^{v_i}\right\} \left(1 + \frac{2(v_i-1)\lambda_i^{v_i}}{v_i} t^{-v_i}\right)^{-1} =$$
$$t^{1-v} \exp\left\{-\frac{1}{2}\left(\frac{t}{\lambda}\right)^{v}\right\} \left\{\sum_{i:v\neq v_i} p_i \frac{\lambda_i^{v_i-1}}{2^{1/v_i}} t^{v-v_i} \Gamma(1/v_i) \exp\left\{-\frac{1}{2}\left(\frac{t}{\lambda_i}\right)^{v_i} + \frac{1}{2}\left(\frac{t}{\lambda}\right)^{v}\right\} \left(1 + \frac{2(v_i-1)\lambda_i^{v_i}}{v_i} t^{-v_i}\right)^{-1} +\right.$$
$$\left.\sum_{i:v=v_i} p_i \frac{\lambda_i^{v_i-1}}{2^{1/v_i}} t^{v-v_i} \Gamma(1/v_i) \exp\left\{-\frac{1}{2}\left(\frac{t}{\lambda_i}\right)^{v_i} + \frac{1}{2}\left(\frac{t}{\lambda}\right)^{v}\right\} \left(1 + \frac{2(v_i-1)\lambda_i^{v_i}}{v_i} t^{-v_i}\right)^{-1}\right\} \sim$$

$$wt^{1-v}\exp\left\{-\frac{1}{2}\left(\frac{t}{\lambda}\right)^{v}\right\}, \tag{5}$$

where $v=\min_{1\leqslant i\leqslant s}\{v_i\}$, λ is the corresponding $\lambda_i(i=1,2,\cdots,s)$ when $v=\min_{1\leqslant i\leqslant s}\{v_i\}$, $w=\sum_{i:v=v_i} p_i\lambda_i^{v_i-1}/(2^{1/v_i}\Gamma(1/v_i))$.

Similarly, we can prove, as $t\to\infty$,

$$B(t)\sim wt^{1-v}\exp\left\{-\frac{1}{2}\left(\frac{t}{\lambda}\right)^{v}\right\}. \tag{6}$$

Therefore, by (5) and (6), we can gain the desired result.

The following corollary provides a Mills' ratio for the MGED.

Corollary 1 $F(t)$ is the cdf of the MGED,

(i) as $t\to\infty$, we have

$$\frac{1-F(t)}{F'(t)}\sim\frac{\lambda w 2^{1+1/v}\Gamma(1/v)}{v}t^{1-v}. \tag{7}$$

(ii) $$1-F(t)=c(t)\exp\left(-\int_{\lambda}^{t}\frac{g(u)}{\tilde{f}(u)}\mathrm{d}u\right).$$

For large enough t, where

$$c(t)\to\lambda^{1-v}w\exp\left(-\frac{1}{2}\right),\ \text{as } t\to\infty,\ \tilde{f}(t)=\frac{2\lambda^{v}}{v}t^{1-v}$$

and

$$g(t)=1+\frac{2(v-1)\lambda^{v}}{v}t^{-v}.$$

Proof (i) (7) follows by Thoerem 3. The proof of (ii) can be found in Peng et al [5]. So we omitted that.

3 Limiting distribution of the MGED

In this section, we consider the limiting distribution of the partial maximum of an independent and identically distributed sequence obeying the MGED and how to choose suitable norming constants such that the distribution of maxima belongs to the domain of attraction of $D(\Lambda)$, where $D(\Lambda)=\exp(-\exp(-x))$.

Theorem 2 Let $\{Z_n, n\geqslant 1\}$ be a sequence of i. i. d. random variables with common distribution $F(x)$ given by (1). Let $M_n=\max(Z_1,Z_2,\cdots,Z_n)$ denote the partial maximum. For $x\in\mathbf{R}$, we have

$$\lim_{n\to\infty}P(M_n\leqslant a_n x+b_n)=\exp(-\exp(-x)),$$

where

$$a_n=\frac{2^{1/v}\lambda}{v\,(\log n)^{1-1/v}}$$

and

$$b_n=2^{1/v}\lambda\,(\log n)^{1/v}+\frac{2^{1/v}\lambda[\log w-((v-1)/v)\log(2\log n)-(v-1)\log\lambda]}{v\,(\log n)^{1-1/v}}.$$

Proof. Combining with Corollary 1(ii) and Theorem 1.6.2 [6-7], we have $F(x)\in D(\Lambda)$. The normalizing constants a_n and b_n can be determined as follows:

Set $u_n=a_n x+b_n$, let us first determine the u_n such that $n(1-F(u_n))\sim\exp(-x)$, as $n\to\infty$.

By Theorem 2, we have

$$1-F(u_n)\sim wu_n^{1-v}\exp\left\{-\frac{1}{2}\left(\frac{u_n}{\lambda}\right)^{v}\right\},$$

hence

$$nwu_n^{1-v}\exp\left\{-\frac{1}{2}\left(\frac{u_n}{\lambda}\right)^{v}+x\right\}\to 1.$$

as $n\to\infty$. So we have

$$\log n+\log w-(v-1)\log u_n-\frac{1}{2}\left(\frac{u_n}{\lambda}\right)^{v}+x\to 0, \tag{8}$$

which implies

$$\frac{u_n^v}{2\lambda^v \log n} \to 1.$$

Taking logarithms, we have

$$v\log u_n - \log 2 - v\log \lambda - \log \log n \to 0,$$

thus

$$\log u_n = \frac{1}{v}(\log 2 + v\log \lambda + \log \log n) + o(1). \quad (9)$$

Putting (9) in (8), we have

$$u_n^v = 2\lambda^v \left[\log n - \frac{v-1}{v}\log(2\log n) - (v-1)\log \lambda + \log w + x + o(1)\right],$$

which implies that

$$u_n = 2^{1/v}\lambda \left[\log n - \frac{v-1}{v}\log(2\log n) - (v-1)\log \lambda + \log w + x + o(1)\right]^{1/v} =$$

$$2^{1/v}\lambda\ (\log n)^{1/v} - \frac{2^{1/v}\lambda}{v\ (\log n)^{1-1/v}} + \left[\frac{v-1}{v}\log(2\log n) + (v-1)\log \lambda - \log w\right] +$$

$$\frac{2^{1/v}\lambda}{v\ (\log n)^{1-1/v}}x + o((\log n)^{1/v-1}) = b_n + a_n x + o((\log n)^{1/v-1}),$$

hence, by Theorem1. 2. 3 [6], the proof is complete.

References:

[1] Nelson D B. Conditional heteroskedasticity in asset returns: A new approach[J]. Econometrica, 1991, 59(2): 347-370.

[2] Mladenovic' P. Extreme values of the sequences of independent random variables with mixed distributions[J]. Matematicki Vesnik, 1999, 51(1/2): 29-37.

[3] Cheng Q, Peng Z. Limiting distributions of maxima on mixed distributions[J]. Journal of Southwest University: Natural Science Edition, 2011, 33(1): 5-10.

[4] Liu B, Peng Z. Extreme value distribution of finite mixed short-tail symmetric distribution[J]. Journal of Southwest University: Natural Science Edition, 2013, 35(1): 85-88.

[5] Peng Z, Tong B, Nadarajah S. Tail behavior of the general error distribution[J]. Communications in Statistics-Theory and Methods, 2009, 38(11): 1884-1892.

[6] Leadbetter M R, Lindgren G, Rootzen H. Extremes and related properties of random sequences and processes[M]. New York: Springer, 1983.

[7] Resnick S I. Extreme value, regular variation, and point processes[M]. New York: Springer-Verlag, 1987.

有限混合广义误差分布的极限分布

黄建文，刘衍民，罗远峰

(遵义师范学院 数学与计算科学学院，贵州 遵义 563002)

摘要:作为正态分布的推广，广义误差分布是在统计中是应用最广泛的分布之一。有限混合分布在现代统计中的整个发展过程中作为一个模型得到了广泛的研究和应用. 这篇文章在广义误差分布的一个重要不等式的基础上研究了有限混合广义误差分布的尾部性质,得到了它的尾部的渐近性质,同时得到了 Mill's 率。然后利用有限混合广义误差分布的尾部渐近性质讨论了同服从该分布的独立随机变量序列最大值的极限分布以及如何选择适当的的规范化常数使得最大值的分布收敛到双指数分布。

关键词:广义误差分布；极值分布；混合分布；尾部性质

(责任编辑 游中胜)