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A Nonlinear Scalarization Characterization of Weakly (C,ε)-Efficient Solutions in Vector Optimization[1]

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Abstract: By algorithmic and theoretical purposes, we often need characterize the whole solution set to a vector optimization problem via scalarization. Because the linear scalarization method often need the generalize convexity assumptions in vector optimization. If without any convexity assumption, we should provide an alternative method by some nonlinear scalarization function . Among the various scalarization techniques, the most widely used two nonlinear functions is Δ function introduced by Hiriart-Urruty and Gerstewitz function introduced by Tammer et al. In recent years, the research of nonlinear scalarization method has become a research focus. Recently, Gutiérrez et al. proposed a new type of efficiency based on co-radiant set which called (C,ε)-efficient solution in vector optimization and gave a necessary condition by the Gerstewitz function. This new notion of efficiency unifies some well-known concepts introduced previously in the literature. In this paper, we establish a new necessary condition via the Δ function for weakly (C,ε)-efficient solution. This can be used to obtain the (C,ε)-efficient solution set by solving the scalar optimization.

Keywords: Weakly (C,ε)-efficient solutions; Nonlinear scalarization; Vector Optimization.

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In recent years, approximate solutions have been playing an important role in vector optimization. One of the most reasons is that approximate solutions can be obtained by using iterative algorithms or heuristic methods. Until now, there are several concepts of approximate solutions. The first one of them is ε-efficiency introduced by Kutateladze [1], and the notion has been used to obtain variational principles, approximate Kuhn-Tucker type conditions, approximate duality theorems, etc. Then, several authors have proposed other ε-efficiency concepts [2-5]. In [6-7], Gutiérrez et al. proposed a new type of efficiency based on co-radiant set which called (C,ε)-efficient solution in vector optimization. This new notion of efficiency unifies some well-known concepts introduced previously in the literature by Kutateladze [1], White [2], Helbig [3], Németh [4] and Tanaka [5]. Moreover, Gao et al. proposed a new properly approximate efficient solution based on the (C,ε)-efficient solution in [8].

Two kinds of nonlinear scalarization functions and their properties are introduced by Zaffaroni [9], Göpfert et al. [10], Tammer [11] and Hiriart-Urruty [12]. In addition, many results via these nonlinear functions have been obtained for approximate solutions. Particularly, In [6], Gutiérrez et al. gave a

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necessary conditions by the nonlinear function introduced by Hiriart-Urruty and a sufficient condition by using a new class of monotone functionals.

Motivated by the works in [6,7,9], we characterize the new (C,ε)-efficient solution by the nonlinear scalarization function studied by Zaffaroni in [9], establish a new necessary condition for weakly (C,ε)-efficient solution.

2 Preliminaries

Throughout this paper, we assume that X and Y be two real topological linear spaces. For a set $A \subset Y$, we denote the topological interior, the topological closure, the boundary and the complement of A by $\operatorname{int} A$, clA, bdA and $Y \setminus A$, respectively. The cone generated by A is defined as $coneA = \bigcup_{\alpha>0} \alpha A$.We say that A is a co-radiant set if A satisfies $\alpha d \in A$ for every $d \in A, \alpha > 1$. Moreover, we say that A is pointed if $A \bigcap (-A) \subset \{0\}$, is solid if $\operatorname{int} A \neq \varnothing$, is proper if A is $\varnothing \neq A \neq Y$. Let $C \subset Y$ be a proper solid pointed co-radiant set and define

$$C(\varepsilon) = \varepsilon C, \forall \varepsilon > 0, C(0) = \bigcup_{\varepsilon>0} C(\varepsilon).$$

Lemma 1[6-7] Assume C be a solid convex set. Then

(i) $C(0) + C(\varepsilon) \subset C(\varepsilon), \forall \varepsilon \geq 0;$

(ii) $C(0)$ is a solid pointed convex cone;

(iii) $\operatorname{int}(clC(\varepsilon)) = \operatorname{int} C(\varepsilon), \forall \varepsilon > 0$.

Remark 1 As C is a proper co-radiant set, it follows that $0 \neq \operatorname{int} C(\varepsilon), \forall \varepsilon > 0$. And $\operatorname{int}(C(0)) = \bigcup_{\varepsilon>0} \varepsilon \operatorname{int}(C) = \bigcup_{\varepsilon>0} \operatorname{int}(C(\varepsilon))$ is an open cone.

Consider the following vector optimization problem:

$$\text{(VP)} \quad \min f(x)$$

$$s.t. \quad x \in S,$$

where $f: X \to Y, S \subset X$ and $S \neq \varnothing$. The preference relation $\leq$ in Y by a proper pointed cone $D \subset Y$ is defined by

$$y, z \in Y, y \leq z \Leftrightarrow y - z \in -D.$$

Definition 1[6-7] Let $\varepsilon \ge 0$. We say that a feasible point $\bar{x} \in S$ is an ε-efficient solution of (VP) with respect to C (or an ε-efficient solution for short) if

$$(f(\bar{x}) - C(\varepsilon)) \cap f(S) \subset \{ f(\bar{x}) \}.$$

Denote by $AE(f, C, \varepsilon)$ the set of (C,ε)-efficient solutions of (VP) .

As C is a solid set, it follows that int C is a nonempty pointed co-radiant set, and we can also consider the set of all ε-efficient solutions of (VP) with respect to $\operatorname{int} C$ (or weakly ε-efficient solutions).

Definition 2[6-7] A feasible point $\bar{x} \in S$ is a weakly ε-efficient solution of (VP) with respect to C if

$$(f(\bar{x}) - \operatorname{int} C(\varepsilon)) \cap f(S) = \varnothing.$$

Denote by $WAE(f, C, \varepsilon)$ the set of weakly (C,ε)-efficient solutions of (VP).

3 Scalarization of Weakly (C,ε)-efficient solutions via $\Delta_{-C(0)}(y)$

In this section, we characterize the (C,ε)-efficient solutions via the nonlinear scalarization function $\Delta_{-C(0)}$ studied by zaffaroni in [4].

For a set $A \in Y$, the function $\Delta_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$\Delta_A(y) = d_A(y) - d_{Y \setminus A}(y),$$

Where $\Delta_{\varnothing}(y) = +\infty, d_A(y) = \inf\limits_{z \in A} \|z - y\|.$

Lemma 2[9] Let A be a proper subset of Y. Then

(i) $\Delta_A(y) < 0$ for every $y \in \operatorname{int} A,$ $\Delta_A(y) = 0$ for every $y \in bdA,$ and $\Delta_A(y) > 0$ for every $y \notin clA$;

(ii) If A is closed, then $A = \{ y | \Delta_A(y) \le 0 \}$;

(iii) If A is convex, then Δ_A is convex ;

(iv) If A is a cone, then Δ_A is positively homogeneous.

Remark 2 If A is a convex cone, then by Lemma 3.1 (iii) and (iv), Δ_A is a sublinear function .

We consider the following scalar problem:

$$\left(P_y\right)\min_{x\in S}\Delta_{-C(0)}(f(x)-y),$$

Where $y\in Y$. We denote the set of ε-minimal solutions and the set of strictly ε-minimal solution of (P_y) by $A\min(\Delta_{-C(0)}(f(x)-y),\varepsilon)$ and $SA\min(\Delta_{-C(0)}(f(x)-y),\varepsilon)$ respectively.

Now, we give a characterize the weakly (C,ε)-efficient solution by the nonlinear scalarization function $\Delta_{-C(0)}(y)$.

Theorem 1 Let C be a proper convex co-radiant set and $\varepsilon\geq 0,\beta=d_{-C}(0)$. Then

$$\bar{x}\in WAE(f,C,\varepsilon)\Rightarrow\bar{x}\in A\min(\Delta_{-C(0)}(f(x)-f(\bar{x})),\varepsilon\beta).$$

Proof Assume that $\bar{x}\in WAE(f,C,\varepsilon)$, i.e.

$$(f(\bar{x})-\operatorname{int}C(\varepsilon))\bigcap f(S)=\varnothing,$$

Then

$$f(x)-f(\bar{x})\notin-\operatorname{int}C(\varepsilon),\forall x\in S. \tag{1}$$

It is clear that

$$C\left(\varepsilon\right)\subset C\left(\varepsilon\right)+\left(C\left(0\right)\bigcup\left\{0\right\}\right).$$

And from Lemma 1(i), we have

$$C(\varepsilon)+(C(0)\cup\{0\})\subset C(\varepsilon).$$

So

$$C(\varepsilon)=C(\varepsilon)+(C(0)\cup\{0\}).$$

Then

$$\operatorname{int}C(\varepsilon)=\operatorname{int}(C(\varepsilon)+(C(0)\cup\{0\}))=\operatorname{int}(C(\varepsilon)+C(0)).$$

Compare with (1), it follows that

$$f(x)-f(\bar{x})\notin-\operatorname{int}(C(\varepsilon)+C(0)),\forall x\in S. \tag{2}$$

Since $C(0)$ is a convex cone and $\operatorname{int}C(0)\neq\varnothing$, from Proposition 2.2 in [13], it follows that

$$\operatorname{int}(C(\varepsilon)+C(0))=\operatorname{int}C(\varepsilon)+\operatorname{int}C(0).$$

So from (2), we can conclude that

$$f(x)-f(\bar{x})+\varepsilon q \notin -\operatorname{int} C(0), \forall x \in S, \forall q \in \operatorname{int} C.$$

On the other hand, $\forall q \in \operatorname{int} C$, we have

$$0 \le \Delta_{-C(0)}(f(x)-f(\bar{x})+\varepsilon q) = d_{-C(0)}(f(x)-f(\bar{x})+\varepsilon q)$$

From the Lemma 1, $C(0)$ is a convex cone, so $\Delta_{-C(0)}$ is a sublinear function. Then

$$0 \le \Delta_{-C(0)}(f(x)-f(\bar{x})+\varepsilon q) \le \Delta_{-C(0)}(f(x)-f(\bar{x}))+\Delta_{-C(0)}(\varepsilon q), \forall q \in \operatorname{int} C \qquad (3)$$

So

$$\Delta_{-C(0)}(f(x)-f(\bar{x}))+\inf_{q\in \operatorname{int} C}\Delta_{-C(0)}(\varepsilon q) \ge 0. \qquad (4)$$

Next, we calculate $\inf_{q\in \operatorname{int} C}\Delta_{-C(0)}(\varepsilon q)$. From the denition of $\Delta_{-C(0)}$, we have

$$\Delta_{-C(0)}(\varepsilon q) = d_{-C(0)}(\varepsilon q) - d_{Y\setminus -C(0)}(\varepsilon q)$$

Because $q \in \operatorname{int} C$, $\varepsilon q \in \operatorname{int} C(\varepsilon) \subset \operatorname{int} C(0)$ and since $C(0)$ is pointed, then $\varepsilon q \notin -\operatorname{int} C(0)$ it follows that

$$\Delta_{-C(0)}(\varepsilon q) = d_{-C(0)}(\varepsilon q).$$

Then

$$\inf_{q\in \operatorname{int} C}\Delta_{-C(0)}(\varepsilon q) = \inf_{q\in \operatorname{int} C}\inf_{\mu\in C(0)}\|\varepsilon q+\mu\|.$$

Furthermore, we can obtain that

$$\inf_{q\in \operatorname{int} C}\inf_{\mu\in C(0)}\|\varepsilon q+\mu\| = \inf_{q\in \operatorname{int} C}\|\varepsilon q\|.$$

In fact, since $\varepsilon q \in C(\varepsilon)$, $\mu \in \operatorname{int} C(0)$ and $C(\varepsilon)+C(0) \subset C(\varepsilon)$, then

$$\varepsilon q+\mu \in C(\varepsilon)+\operatorname{int} C(0) \subset C(\varepsilon).$$

Then we have

$$\inf_{\mu\in C(0)}\|\varepsilon q+\mu\| \ge \inf_{q\in \operatorname{int} C}\|\varepsilon q\|, \forall q \in \operatorname{int} C.$$

Moreover, for any given $\varepsilon_1 > 0$, there exists $q_0 \in \operatorname{int} C$, s.t.

$$\|\varepsilon q_0\| < \inf_{q\in \operatorname{int} C}\|\varepsilon q\| + \varepsilon_1,$$

Therefore, there exist $\bar{q} \in C, \bar{\mu} \in C(0)\bigcup\{0\}$ s.t. $\varepsilon q_0 = \varepsilon\bar{q}+\bar{\mu}$. Then

$$\inf_{\mu\in C(0)}\|\varepsilon q+\mu\|\le\|\varepsilon\overline{q}+\overline{\mu}\|=\|\varepsilon q_0\|<\inf_{q\in\operatorname{int}C}\|\varepsilon q\|+\varepsilon_1,$$

It follows that

$$\inf_{q\in\operatorname{int}C}\inf_{\mu\in C(0)}\|\varepsilon q+\mu\|=\inf_{q\in\operatorname{int}C}\|\varepsilon q\|=\varepsilon\inf_{q\in\operatorname{int}C}\|q\|=\varepsilon\inf_{q\in C}\|q\|=\varepsilon d_{-C}(0)=\varepsilon\beta.$$

From (3) and (4), we conclude that

$$\Delta_{-C(0)}(f(x)-f(\overline{x}))\ge 0\ge\Delta_{-C(0)}(f(\overline{x})-f(\overline{x}))-\varepsilon\beta$$

i.e.,

$$\overline{x}\in AMin(f(x)-f(\overline{x}),\varepsilon\beta).$$

In the following example, we illustrate the above theorem 1.

Example 1 Let $X=Y=R^2$, $C=\{(x_1,x_2)\,|\,x_1+x_2\ge 1,x_1\ge 0,x_2\ge 0\}$,

$S=\{(x_1,x_2)\,|\,x_1+x_2\le 1,x_1\ge 0,x_2\ge 0\}$. The map $f:X\to Y$ is defined as $f(x)=x$.

Let $\overline{x}=(0,0)$, we can verify that $(f(\overline{x})-\operatorname{int}C(\varepsilon))\bigcap f(S)=\varnothing$, $\Delta_{-C(0)}(f(x)-f(\overline{x}))\ge 0$ and

$\Delta_{-C(0)}(f(\overline{x})-f(\overline{x}))=0$, $\beta=\dfrac{\sqrt{2}}{2}$.

Therefore $$\Delta_{-C(0)}(f(x)-f(\overline{x}))\ge\Delta_{-C(0)}(f(\overline{x})-f(\overline{x}))-\varepsilon\beta.$$

Hence $$\overline{x}\in AMin(f(x)-f(\overline{x}),\varepsilon\beta).$$

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运筹学与控制论

向量优化问题(C,ε)弱有效的一个非线性标量化性质

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摘要：非线性标量化研究正在成为向量优化领域中的研究热点之一。有文献在co-radiant集的基础上提出了一种新的(C,ε)－弱有效解并利用Gerstewitz泛函给出了这种新的一个必要条件，它统一了几种经典的近似解。笔者利用 Hiriart-Urruty 等人提出的非线性标量化函数，建立了(C,ε)－弱有效解的一个必要条件。

关键词：(C,ε)－弱有效解；非线性标量化；向量优化.