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Notes on Characterizations and Applications of Semi-prequasi-invexity*

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Abstract: [**Purposes**] The semi-prequasi-invexity and its applications are further investigated. [**Methods**] With assumptions B1, B2, and denseness results. [**Findings**] Firstly, under more weaker assumptions, some new characterizations of semi-prequasi-invexity are obtained. Then, the optimality conditions of semi-prequasi-invex type mathematical programming problems are given in the case of without constraints and with inequality constraints, respectively. Finally, several applicable results of semi-prequasi-invexity in multiobjective optimization problem are gained, and examples are also shown to illustrate the results. [**Conclusions**] The obtained results extend and improve some latest literatures.

Keywords: characterizations; semicontinuity; semi-prequasi-invex functions; nonlinear programming; multiobjective optimization problem

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Convexity and generalized convexity play an essential role in optimization theory. Therefore, seeking some practical criteria for convexity or generalized convexity is especially crucial. In recent decades, there has been a multitude of compositions exploring on this subject[1-11]. Particularly, [1-2] established the characterizations for the classical invexity. Yang et al.[3] founded the properties of prequasi-invex functions under a semicontinuity condition. Luo et al.[4-5] improved the results in [3] under weaker assumptions. Yang and Li[6-7] presented some properties of preinvex functions and semistrictly preinvex functions. Then, two significant generalizations of convex functions are the so-called semi-preinvex function and G-preinvex functions were introduced by Yang[8] and Antczak[12]. And then, Luo et al.[9] discussed the relationships between G-preinvex functions and semistrictly G-preinvex functions. Very recently, Zhao et al.[10] obtained some properties and important char-

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acterizations of semi-prequasi-invexity. Yang[11] established an important property regarding to condition C for preinvexity.

Motivated by the works of [8-11], in this paper, we give some new results for semi-prequasi-invexity. Firstly, under weaker assumptions, we provide some new characterizations of semi-prequasi-invexity, which improve [10]. Then, we discuss the applications of semi-prequasi-invex type functions without constraints and with inequality constraints nonlinear programming, respectively. Finally, the applications of semi-prequasi-invex type functions in multiobjective optimization problem are studied, and some examples are also given to illustrate the results.

1 Preliminaries

Now let us recall some fundamental concepts about semi-prequasi-invexity.

Definition 1 A set $K\subseteq\mathbf{R}^n$ is said to be semi-connected if there exists a vector-valued function $\eta: K\times K\times[0,1]\to K$, such that $\boldsymbol{x},\boldsymbol{y}\in K,\lambda\in[0,1]\Rightarrow\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)\in K$.

Example 1 This example illustrates the existence of semi-connected set. Let $K=[-1,0)\cup(0,1]$, and

$$\eta(x,y,\lambda)=\begin{cases}x-y, 1\geqslant x>0, 1\geqslant y>0\\ x-y, -1\leqslant x<0, -1\leqslant y<0\\ -\dfrac{y}{2}-\dfrac{\lambda}{2}, 1\geqslant x>0, -1\leqslant y<0\\ -\dfrac{y}{2}+\dfrac{\lambda}{2}, -1\leqslant x<0, 1\geqslant y>0\end{cases}$$

. Then, it is easy to verify that

$$y+\lambda\eta(x,y,\lambda)\in K, \forall x,y\in K,\lambda\in[0,1],$$

This is, K is a semi-connected set with respect to $\eta(x,y,\lambda)$.

The following class of semi-prequasi-invex functions was introduced by Yang[8].

Definition 2 Let $K\subseteq\mathbf{R}^n$ be a semi-connected set with respect to η. Let $\eta: K\times K\times[0,1]\to K$. We say that $f: K\to\mathbf{R}$ is semi-prequasi-invex if,

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},\ \forall\boldsymbol{x},\boldsymbol{y}\in K,\lambda\in[0,1].$$

We give the following example to illustrate the existence of semi-preqsi-invex functions.

Example 2 Let $K=\mathbf{R}$, $f(x)=\begin{cases}1, x\geqslant 0\\ 0, x<0\end{cases}$, $\eta(x,y,\lambda)=\begin{cases}x+y-\lambda, x\geqslant 0, y\geqslant 0 \text{ or } x<0, y<0\\ x-y-\lambda, x\geqslant 0, y<0 \text{ or } x<0, y\geqslant 0\end{cases}$. Then, it is easy to check that K is a semi-connected set with respect to η, and f is a semi-prequai-invex function.

Remark 1 It is clear that a prequasi-invex function is a semi-prequasi-invex function when $\eta(\boldsymbol{x},\boldsymbol{y},\lambda)=\eta(\boldsymbol{x},\boldsymbol{y})$. But the converse is not true.

Definition 3 Let $K\subseteq\mathbf{R}^n$ be a semi-connected set with respect to $\eta: K\times K\times[0,1]\to K$. Let $f: K\to\mathbf{R}$. We say that f is semistrictly semi-prequasi-invex if,

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}, \forall\boldsymbol{x},\boldsymbol{y}\in K, f(\boldsymbol{x})\neq f(\boldsymbol{y}),\lambda\in(0,1).$$

Definition 4 Let $K\subseteq\mathbf{R}^n$ be a semi-connected set with respect to $\eta: K\times K\times[0,1]\to K$. Let $f: K\to\mathbf{R}$. We say that f is strictly semi-prequasi-invex if,

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},\ \forall\boldsymbol{x},\boldsymbol{y}\in K,\boldsymbol{x}\neq\boldsymbol{y},\lambda\in(0,1).$$

Remark 2 It is apparent that strict semi-prequasi-invexity implies semistrict semi-prequasi-invexity.

Example 3 This example illustrates that a semistrictly semi-prequasi-invex function is unnecessarily a semi-prequsai-invex function and a strictly semi-prequsai-invex function. Let $K=\mathbf{R}$,

$$f(x)=\begin{cases}-\frac{1}{3}|x|, & |x|\leqslant 1\\ -\frac{1}{3}, & |x|\geqslant 1\end{cases},\eta(x,y,\lambda)=\begin{cases}x-y+\lambda, x\geqslant 0, y\geqslant 0\\ x-y-\lambda, x\leqslant 0, y\leqslant 0\\ x-y+\lambda, x>1, y<-1\\ x-y-\lambda, x<-1, y>1\\ y-x+\lambda, -1\leqslant x\leqslant 0, y\geqslant 0\\ y-x-\lambda, x\geqslant 0, -1\leqslant y\leqslant 0\\ y-x-\lambda, 0\leqslant x\leqslant 1, y\leqslant 0\\ y-x+\lambda, x\leqslant 0, 0\leqslant y\leqslant 1\end{cases}.$$

It is obvious that K is a semi-connected set and we can check that f is a semistrictly semi-prequasi-invex function with respect to η on K. However, let $x=4, y=-4, \lambda=\frac{1}{2}$, we have

$$f(y+\lambda\eta(x,y,\lambda))=f\left(-4+\frac{1}{2}\left(4-(-4)+\frac{1}{2}\right)\right)=f\left(\frac{1}{4}\right)=-\frac{1}{12}>\max\{f(4),f(-4)\}=-\frac{1}{3}.$$

Thus, f is neither a semi-prequasi-invex function for the same η, nor a strictly semi-prequasi-invex function for the same η.

In the sequel, we give a basic result on semi-prequasi-invex type function.

Theorem 1 Let K be a nonempty semi-connected set in $\mathbf{R}^n$ with respect to $\eta: K\times K\times[0,1]\rightarrow K$, and let $f: K\rightarrow\mathbf{R}$ be a semistrictly semi-prequasi-invex function for the same η, and $g: I\rightarrow\mathbf{R}$ be a strictly increasing function, where range$(f)\subseteq I$. Then, the composite function $g(f)$ is a semistricly semi-prequasi-invex function on K.

Proof For any $\boldsymbol{x},\boldsymbol{y}\in K, \lambda\in(0,1)$, if $g(f(\boldsymbol{x}))\neq g(f(\boldsymbol{y}))$, then, $f(\boldsymbol{x})\neq f(\boldsymbol{y})$. Since f is a semistricly semi-prequasi-invex function, we have $f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$. From the strictly increasing property of g, we have

$$g[f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))]<g(\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\})=\max\{g(f(\boldsymbol{x})),g(f(\boldsymbol{y}))\}.$$

Hence, $g(f)$ is a semistricly semi-prequasi-invex function on K.

Remark 3 In Theorem 3.2 of [8], the assumption of convexity for $g: I\rightarrow\mathbf{R}$ is required, while it's not required in Theorem 1, and it's extended to semi-prequasi-invexity case.

In researching the characterizations for semi-prequasi-invex functions, we demand the following conditions, which have been presented by Zhao[10].

Condition B1 $\eta(\boldsymbol{x},\boldsymbol{y},\lambda)$ is said to satisfy Condition B1 if, for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\alpha,\lambda_1,\lambda_2\in[0,1]$,

$$\boldsymbol{y}+((1-\alpha)\lambda_1+\alpha\lambda_2)\eta(\boldsymbol{x},\boldsymbol{y},(1-\alpha)\lambda_1+\alpha\lambda_2)=z_1+\alpha\eta(z_2,z_1,\alpha),$$

where $z_1=\boldsymbol{y}+\lambda_1\eta(\boldsymbol{x},\boldsymbol{y},\lambda_1)$ and $z_2=\boldsymbol{y}+\lambda_2\eta(\boldsymbol{x},\boldsymbol{y},\lambda_2)$.

Condition B2 $\eta(\boldsymbol{x},\boldsymbol{y},\lambda)$ is said to satisfy Condition B2 if, for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\alpha,\lambda\in[0,1]$,

$$\boldsymbol{y}+((1-\alpha)\lambda+\alpha)\eta(\boldsymbol{x},\boldsymbol{y},(1-\alpha)\lambda+\alpha)=z+\alpha\eta(\boldsymbol{x},\boldsymbol{z},\alpha),$$

where $\boldsymbol{z}=\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)$.

Condition B3 Let K be semi-connected set with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\lambda)$, $f(\boldsymbol{x})$ is said to satisfy Condition B3 if, for all $\boldsymbol{x},\boldsymbol{y}\in K$, $f(\boldsymbol{y}+\eta(\boldsymbol{x},\boldsymbol{y},1))\leqslant f(\boldsymbol{x})$.

Example 4 This example illustrates the existence of $\eta(x,y,\lambda)$, which satisfies Condition B1 and B2 on the semi-connected set K. Let $K=\mathbf{R}$ and $\eta(x,y,\lambda)=x-y$.

It is easy to see that K is a semi-connected set with respect to $\eta(x,y,\lambda)$, and from the definition of Condition B1 and B2, we can verify that $\eta(x,y,\lambda)$ satisfies Conditions B1 and B2 on the semi-connected set K.

2 Some new characterizations of semi-prequasi-invex type functions

Throughout this section, let K be a semi-connected set with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\lambda)$ in $\mathbf{R}^n$.

Lemma 1[10]　Suppose that $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ satisfies Condition B1, $f(\boldsymbol{x})$ satisfies Condition B3, and if there exists $\alpha\in(0,1)$, such that $f(\boldsymbol{y}+\alpha\eta(\boldsymbol{x},\boldsymbol{y},\alpha))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$, $\forall\,\boldsymbol{x},\boldsymbol{y}\in K$, then, the set A defined below is dense in the unit interval $[0,1]$. $A=\{\lambda\in[0,1]:f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},\forall\,\boldsymbol{x},\boldsymbol{y}\in K\}$.

Now, we improve Lemma 1 by Lemma 2 as follows.

Lemma 2　Suppose that $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ satisfies Condition B1, $f(\boldsymbol{x})$ satisfies Condition B3, and for each pair of $\boldsymbol{x},\boldsymbol{y}\in K$, if there exists $\alpha_{x,y}\in(0,1)$, such that $f(\boldsymbol{y}+\alpha_{x,y}\eta(\boldsymbol{x},\boldsymbol{y},\alpha_{x,y}))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$, then, the set A is dense in the unit interval $[0,1]$, $A=\{\lambda\in[0,1]:f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},\forall\,\boldsymbol{x},\boldsymbol{y}\in K\}$.

Proof　It is explicit from Condition B3 that $0,1\in A$. Below, on the contradiction. Suppose that A is not dense in $[0,1]$, this is, there exist $\lambda_0\in(0,1)$ and a neighborhood of λ_0, denoted by $N(\lambda_0)$, such that

$$N(\lambda_0)\cap A=\varnothing. \tag{1}$$

Since $0,1\in A$, it is clear that $\{\lambda\in A:\lambda\geqslant\lambda_0\}\neq\varnothing$, $\{\lambda\in A:\lambda\leqslant\lambda_0\}\neq\varnothing$. Let $\lambda_1=\inf\{\lambda\in A:\lambda\geqslant\lambda_0\}$, $\lambda_2=\sup\{\lambda\in A:\lambda\leqslant\lambda_0\}$. By (1), we have $0\leqslant\lambda_2<\lambda_1\leqslant 1$. Since, for each pair of $\boldsymbol{x},\boldsymbol{y}\in K$, $\max\{\alpha_{x,y},(1-\alpha_{x,y})\}\in(0,1)$, there always exist $\mu_1,\mu_2\in A$ with $\mu_1\geqslant\lambda_1$ and $\mu_2\leqslant\lambda_2$, such that

$$(\max\{\alpha_{x,y},(1-\alpha_{x,y})\})(\mu_1-\mu_2)<\lambda_1-\lambda_2. \tag{2}$$

For each pair of $\boldsymbol{x},\boldsymbol{y}\in K$, denote $z_1=\boldsymbol{y}+\mu_1\eta(\boldsymbol{x},\boldsymbol{y},\mu_1)$, $z_2=\boldsymbol{y}+\mu_2\eta(\boldsymbol{x},\boldsymbol{y},\mu_2)$. Let

$$\bar{\lambda}=\alpha_{z_1,z_2}\mu_1+(1-\alpha_{z_1,z_2})\mu_2\in(\mu_2,\mu_1).$$

By Condition B1, we have $\boldsymbol{y}+\bar{\lambda}\eta(\boldsymbol{x},\boldsymbol{y},\bar{\lambda})=z_2+\alpha_{z_1,z_2}\eta(z_1,z_2,\alpha_{z_1,z_2})$, which implies that,

$$f(\boldsymbol{y}+\bar{\lambda}\eta(\boldsymbol{x},\boldsymbol{y},\bar{\lambda}))=f(z_2+\alpha_{z_1,z_2}\eta(z_1,z_2,\alpha_{z_1,z_2}))\leqslant\max\{f(z_1),f(z_2)\}\leqslant$$
$$\max\{\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}\}\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},$$

which means $\bar{\lambda}\in A$. We divide the proof by two cases of $\bar{\lambda}$.

If $\bar{\lambda}\geqslant\lambda_0$, then by (2), we have $\bar{\lambda}-\mu_2=\alpha_{z_1,z_2}(\mu_1-\mu_2)<\lambda_1-\lambda_2$. It follows that $\bar{\lambda}<\lambda_1$, which contradicts with the definition of λ_1.

If $\bar{\lambda}<\lambda_0$, then we can conclude again from (2) that $\bar{\lambda}>\lambda_2$, which contradicts with the definition of λ_2. Hence, the result is obtained.

Remark 4　In Lemma 1, a uniform $\lambda\in(0,1)$ is needed, while in Lemma 2 this condition has been weakened to a great extent.

Theorem 2[10]　Let $f(\boldsymbol{x})$ be upper semi-continuous on K and satisfy Condition B3. Let $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ satisfy Conditions B1, B2, and $\lim\limits_{\theta\to 0}\theta\eta(\boldsymbol{x},\boldsymbol{y},\theta)=0$ for all $\boldsymbol{x},\boldsymbol{y}\in K$. Then, if there exists $\lambda\in(0,1)$, such that

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},\ \forall\,\boldsymbol{x},\boldsymbol{y}\in K,$$

then $f(\boldsymbol{x})$ is semi-prequasi-invex with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K.

The above Theorem 2 can be improved as follows.

Theorem 3　Let $f(\boldsymbol{x})$ be upper semi-continuous on K and satisfies Condition B3. Let $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ satisfies Conditions B1, B2, and $\lim\limits_{\theta\downarrow 0}\theta\eta(\boldsymbol{x},\boldsymbol{y},\theta)=0$ for all $\boldsymbol{x},\boldsymbol{y}\in K$. For each pair of $\boldsymbol{x},\boldsymbol{y}\in K$, if there exists $\lambda_{x,y}\in(0,1)$, such that $f(\boldsymbol{y}+\lambda_{x,y}\eta(\boldsymbol{x},\boldsymbol{y},\lambda_{x,y}))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$, then $f(\boldsymbol{x})$ is semi-prequasi-invex with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K.

Proof　By contradiction, we assume that $f(\boldsymbol{x})$ is not semi-prequasi-invex with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K. Then, there exist $\boldsymbol{x},\boldsymbol{y}\in K$ and $\bar{\alpha}\in(0,1)$, such that

$$f(\boldsymbol{y}+\bar{\alpha}\eta(\boldsymbol{x},\boldsymbol{y},\bar{\alpha}))>\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}. \tag{3}$$

It is evident by Lemma 2 that the set A is dense in $[0,1]$,

$$A=\{\alpha\in[0,1]:f(\boldsymbol{y}+\alpha\eta(\boldsymbol{x},\boldsymbol{y},\alpha))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},\forall\,\boldsymbol{x},\boldsymbol{y}\in K\}.$$

Consequently, there exists a sequence $\{\alpha_n\}\subseteq A\cap(0,1)$ with $\alpha_n<\bar{\alpha}$ such that $\lim\limits_{n\to\infty}\alpha_n\to\bar{\alpha}$. Let $\boldsymbol{z}=\boldsymbol{y}+\bar{\alpha}\eta(\boldsymbol{x},\boldsymbol{y},\bar{\alpha})$, and

$\boldsymbol{y}_n=\boldsymbol{y}+\left(\frac{\bar{\alpha}-\alpha_n}{1-\alpha_n}\right)\eta\left(\boldsymbol{x},\boldsymbol{y},\frac{\bar{\alpha}-\alpha_n}{1-\alpha_n}\right)$.

By $\lim_{\theta\downarrow 0}\theta\eta(\boldsymbol{x},\boldsymbol{y},\theta)=0$, we have $\lim_{n\to\infty}\boldsymbol{y}_n\to\boldsymbol{y}$. Since K is semi-connected with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$, it follows from $\frac{\bar{\alpha}-\alpha_n}{1-\alpha_n}\in(0,1)$ that $\boldsymbol{y}_n\in K$. Again, from Condition B2 we obtain

$$\boldsymbol{z}=\boldsymbol{y}+\bar{\alpha}\eta(\boldsymbol{x},\boldsymbol{y},\bar{\alpha})=\boldsymbol{y}+\left(\alpha_n+(1-\alpha_n)\frac{\bar{\alpha}-\alpha_n}{1-\alpha_n}\right)\eta\left(\boldsymbol{x},\boldsymbol{y},\alpha_n+(1-\alpha_n)\frac{\bar{\alpha}-\alpha_n}{1-\alpha_n}\right)=\boldsymbol{y}_n+\alpha_n\eta(\boldsymbol{x},\boldsymbol{y}_n,\alpha_n).$$

By the upper semi-continuity property of $f(\boldsymbol{x})$ on K, there exists $N>0$ such that, for all $n>N$,

$$f(\boldsymbol{y}_n)\leqslant f(\boldsymbol{y})+\varepsilon,\ \forall\,\varepsilon>0. \tag{4}$$

By $\alpha_n\in A$ and (4), it follow that, for all $n>N$,

$$f(\boldsymbol{z})=f(\boldsymbol{y}+\bar{\alpha}\eta(\boldsymbol{x},\boldsymbol{y},\bar{\alpha}))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y}_n)\}\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})+\varepsilon\}.$$

Since ε is arbitrary, then $f(\boldsymbol{z})=f(\boldsymbol{y}+\bar{\alpha}\eta(\boldsymbol{x},\boldsymbol{y},\bar{\alpha}))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$, which contradicts (3). This completes the proof.

Theorem 4[10] Let $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ satisfy Conditions B1, B2. Let $f(\boldsymbol{x})$ be semistrictly semi-prequasi-invex with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K, and satisfy Condition B3. Then, if there exists $\alpha\in(0,1)$, such that for all $\boldsymbol{x},\boldsymbol{y}\in K$ with $\boldsymbol{x}\neq\boldsymbol{y}$, $f(\boldsymbol{y}+\alpha\eta(\boldsymbol{x},\boldsymbol{y},\alpha))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$, then $f(\boldsymbol{x})$ is strictly semi-prequasi-invex with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K.

Now, we improve the above theorem, as follows.

Theorem 5 Let $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ satisfy Conditions B1, B2. Let $f(\boldsymbol{x})$ be semistrictly semi-prequasi-invex with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K. Then, for each pair $\boldsymbol{x},\boldsymbol{y}\in K$ with $\boldsymbol{x}\neq\boldsymbol{y}$, if there exists $\alpha_{x,y}\in(0,1)$ such that,

$$f(\boldsymbol{y}+\alpha_{x,y}\eta(\boldsymbol{x},\boldsymbol{y},\alpha_{x,y}))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}, \tag{5}$$

then $f(\boldsymbol{x})$ is strictly semi-prequasi-invex with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K.

Proof Suppose there exist $\boldsymbol{x},\boldsymbol{y}\in K,\boldsymbol{x}\neq\boldsymbol{y},\lambda\in(0,1)$ such that $f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\geqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$. Denoted $\boldsymbol{z}=\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)$, then

$$f(\boldsymbol{z})\geqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}. \tag{6}$$

If $f(\boldsymbol{x})\neq f(\boldsymbol{y})$, by the semistrict semi-prequasi-invexity of f, we have $f(\boldsymbol{z})<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$, which contradicts (6). Thus, we obtain $f(\boldsymbol{x})=f(\boldsymbol{y})$, and also by (6), we have

$$f(\boldsymbol{z})\geqslant f(\boldsymbol{x})=f(\boldsymbol{y}). \tag{7}$$

Note that the pair $\boldsymbol{x},\boldsymbol{y}$ is distinct. From (5), there exists $\alpha_{x,y}\in(0,1)$, such that

$$f(\boldsymbol{y}+\alpha_{x,y}\eta(\boldsymbol{x},\boldsymbol{y},\alpha_{x,y}))<f(\boldsymbol{x})=f(\boldsymbol{y}). \tag{8}$$

Denote $\bar{\boldsymbol{y}}=\boldsymbol{y}+\alpha_{x,y}\eta(\boldsymbol{x},\boldsymbol{y},\alpha_{x,y})$.

If $\lambda<\alpha_{x,y}$, let $\mu=(\alpha_{x,y}-\lambda)/\alpha_{x,y}$, then $\mu\in(0,1)$, according to Condition B1 we have

$$\bar{\boldsymbol{y}}+\mu\eta(\boldsymbol{y},\bar{\boldsymbol{y}},\mu)=\boldsymbol{y}+((1-\mu)\alpha_{x,y})\eta(\boldsymbol{x},\boldsymbol{y},(1-\mu)\alpha_{x,y})=\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)=\boldsymbol{z},$$

which, together with (8) and f is semistrictly semi-prequasi-invex function with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K, yields $f(\boldsymbol{z})=f(\bar{\boldsymbol{y}}+\mu\eta(\boldsymbol{y},\bar{\boldsymbol{y}},\mu))<\max\{f(\bar{\boldsymbol{y}}),f(\boldsymbol{y})\}=f(\boldsymbol{y})$. This contradicts (7).

If $\lambda>\alpha_{x,y}$, define $\upsilon=(\lambda-\alpha_{x,y})/(1-\alpha_{x,y})$, so $\upsilon\in(0,1)$, from Condition B2 we obtain

$$\bar{\boldsymbol{y}}+\upsilon\eta(\boldsymbol{x},\bar{\boldsymbol{y}},\upsilon)=\boldsymbol{y}+((1-\upsilon)\alpha_{x,y}+\upsilon)\eta(\boldsymbol{x},\boldsymbol{y},(1-\upsilon)\alpha_{x,y}+\upsilon)=\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)=\boldsymbol{z}.$$

(8) and f is semistrictly semi-prequasi-invex function with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K, implies that

$$f(\boldsymbol{z})=f(\bar{\boldsymbol{y}}+\upsilon\eta(\boldsymbol{x},\bar{\boldsymbol{y}},\upsilon))<\max\{f(\bar{\boldsymbol{y}}),f(\boldsymbol{x})\}=f(\boldsymbol{x}),$$

which contradicts (7). This completes the proof.

Remark 5 A uniform $\alpha\in(0,1)$ is needed in Theorem 4[10], while in Theorem 5 it is weakened to a great extent. Moreover, the Condition B3 is deleted here.

Example 5 This example illustrates that, in Theorem 5, for each pair $\boldsymbol{x},\boldsymbol{y}\in K$ with $\boldsymbol{x}\neq\boldsymbol{y}$ for all $\alpha_{x,y}\in(0,1)$

if (5) does not hold, then the result maybe not true. Let $K=[0,+\infty)$, $\eta(x,y,\lambda)=x-y$, and $f(x)=\begin{cases}-x, x\leqslant 1\\ -1, x\geqslant 1\end{cases}$.

It is effortless to see that K is a semi-connected set with respect to $\eta(x,y,\lambda)$, and from the definition of Condition B1 and B2, we can verify that $\eta(x,y,\lambda)$ satisfies Conditions B1 and B2 on the semi-connected set K. Then, according to Definition 3, we can cheek that $f(x)$ is a semistrict semi-prequasi-invex function. However, for arbitrarily $x,y\geqslant 1, x\neq y, \alpha_{x,y}\in(0,1)$, we have $f(y+\alpha_{x,y}\eta(x,y,\alpha_{x,y}))\not<\max\{f(x),f(y)\}$.

Letting $x=2, y=3, \lambda\in(0,1)$, we have $f(3+\lambda\eta(2,3,\lambda))=f(3-\lambda)=-1=f(x)=f(y)$. Thus, $f(x)$ is not a strict semi-prequasi-invex function with respect to the same $\eta(x,y,\lambda)$ on K.

3 Applications in two kinds of nonlinear programming problems

The applications of semi-prequasi-invex type functions without constraints and with inequality constraints nonlinear programming will be discussed, respectively. Now, we first consider the following nonlinear programming problem without constraints.

$$(\mathrm{P}_1): \min f(\boldsymbol{x}),$$
$$\boldsymbol{x}\in K,$$

where K is a subset of $\mathbf{R}^n$, $f(\boldsymbol{x})$ is a real-valued function on K.

Theorem 6　For (P_1), suppose that $K\subseteq\mathbf{R}^n$ be a semi-connected set with respect to $\eta: K\times K\times[0,1]\rightarrow K$. If f is semistrictly semi-prequasi-invex and semi-prequasi-invex function with respect to η, and $\lim_{\lambda\downarrow 0}\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)=0$, then, (i) any local efficient solution of (P_1) is a global efficient solution of (P_1); (ii) the solution set of (P_1) is a semi-connected set with respect to the same η.

Proof　(i) Assume on the contrary that there exists $\boldsymbol{y}\in K$ such that, $\boldsymbol{y}$ is a local efficient solution of (P_1), but is not a global efficient solution of (P_1). Then, there exists $\boldsymbol{x}\in K$ such that $f(\boldsymbol{x})<f(\boldsymbol{y})$.

And from the semistrictly semi-prequasi-invexity of f, we have

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}=f(\boldsymbol{y}),\ \forall\lambda\in(0,1). \tag{9}$$

Since λ can be arbitrary small, (9) implies that $\boldsymbol{y}$ is not a local efficient solution of (P_1), which is a contradiction.

(ii) Let $\alpha=\inf_{x\in K}f(\boldsymbol{x})$, and $S=\{\boldsymbol{x}\in K: f(\boldsymbol{x})=\alpha\}$. Now, for any $\boldsymbol{x},\boldsymbol{y}\in S$, by the semi-prequasi-invexity of f on K with respect to η, we have

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}=\alpha,\ \forall\lambda\in[0,1]\Rightarrow\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)\in S,\ \forall\lambda\in[0,1].$$

Hence, the solution set of (P_1) is a semi-connected set with respect to η.

Theorem 7　Let $f(\boldsymbol{x})$ be strictly semi-prequasi-invex with respect to the vector-valued function η, and $\lim_{\lambda\downarrow 0}\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)=0$. Then, the solution of (P_1) is unique.

Proof　On the contrary, let $\boldsymbol{y}$ be a solution of (P_1), if the solution of (P_1) is not unique. Then, there exists $\boldsymbol{x}\in K$ such that $\boldsymbol{x}\neq\boldsymbol{y}$ and $f(\boldsymbol{x})=f(\boldsymbol{y})$. Since, $f(\boldsymbol{x})$ be strictly semi-prequasi-invex function with respect to the vector-valued function η, we obtain

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))<f(\boldsymbol{y}),\ \forall\lambda\in(0,1),$$

which implies that y is not a solution of (P_1), this is a contradiction.

We give the following example to illustrate the correctness of the Theorem 6.

Example 6　Let $K=[-1,1]$, $f(x)=x^2$ and $\eta(x,y,\lambda)=\lambda x-y$.

Then, it is easy to affirm that K is a semi-connected set with respect to η, and $\lim_{\lambda\downarrow 0}\lambda\eta(x,y,\lambda)=0$, and $f(x)$ is a strictly semi-prequai-invex function with respect to the same η on K. Obviously, we can see that the solution set of (P_1) is a singleton set $\{0\}$, that is, the solution of (P_1) is $x=0$, and it is unique.

Now, we consider the following nonlinear programming problem with inequality constraints.

$$(P_2):\ \min f(\boldsymbol{x}),$$
$$g_i(\boldsymbol{x})\leqslant 0,\ i\in J=\{1,\cdots,m\},\ \boldsymbol{x}\in K,$$

Where K is a subset of $\mathbf{R}^n$, $f, g_i(i\in J)$ are real-valued functions on K. And $D=\{\boldsymbol{x}\in K\mid g_i(\boldsymbol{x})\leqslant 0,\ i\in J\}$ denotes the feasible set of (P_2).

Theorem 8 For (P_2), suppose that $K\subseteq\mathbf{R}^n$ be a semi-connected set with respect to $\eta: K\times K\times[0,1]\to K$, and $f, g_i(i\in J)$ are semi-prequasi-invex functions with respect to the same η. Then, the feasible set D and optimal solution set of (P_2) is a semi-connected set with respect to the same η.

Proof (a) suppose $\boldsymbol{x},\boldsymbol{y}\in D$, we have $\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)\in K$, and $g_i(\boldsymbol{x})\leqslant 0, g_i(\boldsymbol{y})\leqslant 0$, $\forall i\in J$. By the semi-prequasi-invexity of $g_i(i\in J)$ on K, we obtain

$$g_i(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\leqslant\max\{g_i(\boldsymbol{x}),g_i(\boldsymbol{y})\}\leqslant 0,\ \forall\lambda\in[0,1],\ i\in J.$$

That is, $\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)\in D$, $\forall\lambda\in[0,1]$.

Thus, the feasible D is a semi-connected set with respect to the same η.

(b) Let C denote the optimal solution set of (P_2), then, for any $\boldsymbol{x},\boldsymbol{y}\in C$, we have $f(\boldsymbol{x})=f(\boldsymbol{y})$. From the result of part (a) we get $\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)\in D$.

Suppose, by contradiction that $\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)\notin C$, it follows that

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))>f(\boldsymbol{x})=f(\boldsymbol{y}).\tag{10}$$

Since $f(\boldsymbol{x})$ is semi-prequasi-invex function with respect to η, we have

$$f(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}=f(\boldsymbol{y}),\ \forall\lambda\in[0,1],$$

which contradicts (10). Thus, the optimal solution set C is a semi-connected set with respect to the same η.

Example 7 This example illustrates the correctness of Theorem 8. Let $K=\mathbf{R}$, $f(x)=\begin{cases}1, x>0\\0, x\leqslant 0\end{cases}$, $g_1(x)=\begin{cases}2, x>0\\0, x\leqslant 0\end{cases}$, $g_2(x)=\begin{cases}3, x>0\\0, x\leqslant 0\end{cases}$ and $\eta(x,y,\lambda)=\begin{cases}x-y+\lambda, x>0, y>0\\x-y-\lambda, x\leqslant 0, y\leqslant 0\\y-x+\lambda, x\leqslant 0, y>0\\y-x-\lambda, x>0, y\leqslant 0\end{cases}$.

Obviously, we can get that K is a semi-connected set with respect to η, and f, g_1, g_2 are semi-prequai-invex functions with respect to the same η on K. Then, it is easy accessible to ascertain that $(-\infty,0]$ is the feasible set and the solution set of (P_2). According to $y+\lambda\eta(x,y,\lambda)=y+\lambda(x-y-\lambda)\in(-\infty,0]$, $\forall\lambda\in[0,1]$, we obtain that the feasible set D and optimal solution set of (P_2) are semi-connected sets with respect to the same η. Thus, the conclusion of Theorem 8 holds.

Theorem 9 For (P_2), suppose that $K\subseteq\mathbf{R}^n$ be a semi-connected set with respect to $\eta: K\times K\times[0,1]\to K$, and $f(\boldsymbol{x})$ is semistrictly semi-prequasi-invex function with $\lim\limits_{\lambda\downarrow 0}\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)=0$. Then, any local efficient solution of (P_2) is a global efficient solution of (P_2).

The proof is similar to the Theorem 6, hence, the proof is omitted.

4 Applications in multiobjective optimization problem

Now, we consider the following multiobjective optimization problem

$$(\mathrm{MP}):\min f(\boldsymbol{x})=(f_1(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x}))^{\mathrm{T}},$$
$$\text{s.t. } \boldsymbol{x}\in K,$$

where $f: K\to\mathbf{R}^m$ is a vector-valued function and $K\subseteq\mathbf{R}^n$ is a semi-connected set with respect to $\eta: K\times K\times$

$[0,1]\to K$.

Let $\mathbf{R}_{+}^{m}=\{x\in\mathbf{R}^m \mid x=(x_1,\cdots,x_m), x_i\geqslant 0, 1\leqslant i\leqslant m\}$, $\mathbf{R}_{++}^{m}=\{x\in\mathbf{R}^m \mid x=(x_1,\cdots,x_m), x_i>0, 1\leqslant i\leqslant m\}$.

In the sequel, we recall the definitions of efficient solution and weakly efficient solution.

Definition 5[4] A point $\bar{x}\in K$ is called a global efficient solution of (MP), if there does not exists any point $y\in K$, such that $f(y)\in f(\bar{x})-\mathbf{R}_{+}^{m}\setminus\{0\}$.

A point $\bar{x}\in K$ is called a local efficient solution of (MP), if there is a neighborhood $N(\bar{x})$ of $\bar{x}$, such that there does not exists any point $y\in K\cap N(\bar{x})$, such that $f(y)\in f(\bar{x})-\mathbf{R}_{+}^{m}\setminus\{0\}$.

Definition 6[4] A point $\bar{x}\in K$ is called a global weakly efficient solution of (MP), if there does not exists any point $y\in K$, such that $f(y)\in f(\bar{x})-\mathbf{R}_{++}^{m}$.

A point $\bar{x}\in K$ is called a local weakly efficient solution of (MP), if there is a neighborhood $N(\bar{x})$ of $\bar{x}$, such that there does not exists any point $y\in K\cap N(\bar{x})$, s. t. $f(y)\in f(\bar{x})-\mathbf{R}_{++}^{m}$.

Then, we give a result on semistrictly semi-prequasi-invex function.

Theorem 10[10] Let $f(x)$ be semi-prequasi-invex with respect to $\eta(x,y,\theta)$ on K. Let $\eta(x,y,\theta)$ satisfy Conditions B1 and B2. Then, if there exists $\alpha\in(0,1)$ such that, for all $x,y\in K$ with $f(x)\neq f(y)$,

$$f(y+\alpha\eta(x,y,\alpha))<\max\{f(x),f(y)\},\ f(y+(1-\alpha)\eta(x,y,1-\alpha))<\max\{f(x),f(y)\},$$

then, $f(x)$ is semistrictly semi-prequasi-invex with respect to $\eta(x,y,\theta)$ on K.

Now, we improve the above theorem, as follows.

Theorem 11 Let $f(x)$ be semi-prequasi-invex with respect to $\eta(x,y,\theta)$ on K. Let $\eta(x,y,\theta)$ satisfy Conditions B1, B2 and $\lim\limits_{\theta\downarrow 0}\theta\eta(x,y,\theta)=0$ for all $x,y\in K$. Then, if there exists $\alpha\in(0,1)$ such that, for all $x,y\in K$ with

$$f(x)\neq f(y),\ f(y+(1-\alpha)\eta(x,y,1-\alpha))<\max\{f(x),f(y)\},$$

then, $f(x)$ is semistrictly semi-prequasi-invex function with respect to $\eta(x,y,\theta)$ on K.

Proof By contradiction, we assume there exist $x,y\in K,\lambda\in(0,1)$ such that, $f(x)\neq f(y)$ and

$$f(y+\lambda\eta(x,y,\lambda))\geqslant\max\{f(x),f(y)\}. \tag{11}$$

Let $z=y+\lambda\eta(x,y,\lambda)$. There are two cases to be considered.

1) We assume that $f(x)<f(y)$. And inequality (11) implies

$$f(z)\geqslant f(y)>f(x). \tag{12}$$

Since $f(x)$ is semi-prequasi-invex function, we obtain

$$f(z)\leqslant\max\{f(x),f(y)\}=f(y),$$

which, together with (12), leads to

$$f(z)=f(y)>f(x). \tag{13}$$

Now, we assert that

$$f(y+\mu\eta(z,y,\mu))=f(z)=f(y),\ \forall\mu\in(0,1). \tag{14}$$

By the way of contradiction, we assume that $\exists\bar{\mu}\in(0,1)$, such that

$$f(y+\bar{\mu}\eta(z,y,\bar{\mu}))<f(z). \tag{15}$$

Denote $\bar{y}=y+\bar{\mu}\eta(z,y,\bar{\mu})$, $\beta_1=(\lambda-\lambda\bar{\mu})/(1-\lambda\bar{\mu})$.

Then, it follows that $0<\beta_1<1$, and from Condition B1, B2, we obtain

$$\begin{aligned}\bar{y}+\beta_1\eta(x,\bar{y},\beta_1)=y+[\beta_1+\bar{\mu}\lambda(1-\beta_1)]\eta(x,y,\beta_1+\bar{\mu}\lambda(1-\beta_1))=\\ y+[\bar{\mu}\lambda+\beta_1(1-\bar{\mu}\lambda)]\eta(x,y,\bar{\mu}\lambda+\beta_1(1-\bar{\mu}\lambda))=y+\lambda\eta(x,y,\lambda)=z,\end{aligned}$$

implying

$$f(z)=f(\bar{y}+\beta_1\eta(x,\bar{y},\beta_1))\leqslant\max\{f(x),f(\bar{y})\}, \tag{16}$$

since $f(x)$ is semi-prequasi-invex. Then, combining (16) and (15) yields $f(z)\leqslant f(x)$, which contradicts (13).

Consequently, (14) holds.

Next, define a sequence $\{\boldsymbol{z}_k\}$ by induction as follows:

$$\boldsymbol{z}_1=\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha),$$
$$\boldsymbol{z}_2=\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{z}_1,\boldsymbol{y},1-\alpha),$$
$$\vdots$$
$$\boldsymbol{z}_k=\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{z}_{k-1},\boldsymbol{y},1-\alpha),\ \forall k\geqslant 2,k\in\mathbf{N}.$$

Then, under the condition that, there exists $\alpha\in(0,1)$, such that for all $\boldsymbol{x},\boldsymbol{y}\in K$ with $f(\boldsymbol{x})\neq f(\boldsymbol{y})$,

$$f(\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},$$

and the assumption $f(\boldsymbol{x})<f(\boldsymbol{y})$, we have

$$f(\boldsymbol{z}_1)=f(\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha))<f(\boldsymbol{y}),$$
$$f(\boldsymbol{z}_2)=f(\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{z}_1,\boldsymbol{y},1-\alpha))<f(\boldsymbol{y}),$$
$$\vdots$$
$$f(\boldsymbol{z}_k)=f(\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{z}_{k-1},\boldsymbol{y},1-\alpha))<f(\boldsymbol{y}),\ \forall k\geqslant 2,k\in\mathbf{N}. \tag{17}$$

From Condition B1, we get $\boldsymbol{z}_k=\boldsymbol{y}+(1-\alpha)^k\eta(\boldsymbol{x},\boldsymbol{y},(1-\alpha)^k),\ \forall k\in\mathbf{N}_+$.

Since $0<\alpha<1$, then $\boldsymbol{z}_k\to\boldsymbol{y}$ as $k\to\infty$. Choose $k_1\in\mathbf{N}$ such that $(1-\alpha)^{k_1}<\lambda$, and denote $\beta=(1-\alpha)^{k_1}/\lambda$, then $0<\beta<1$. Then, from Condition B1, we obtain

$$\boldsymbol{y}+\beta\eta(\boldsymbol{z},\boldsymbol{y},\beta)=\boldsymbol{y}+\lambda\beta\eta(\boldsymbol{x},\boldsymbol{y},\lambda\beta)=y+(1-\alpha)^{k_1}\eta(\boldsymbol{x},\boldsymbol{y},(1-\alpha)^{k_1})=\boldsymbol{z}_{k_1},$$

which, together with (17), yields

$$f(\boldsymbol{y}+\beta\eta(\boldsymbol{z},\boldsymbol{y},\beta))=f(\boldsymbol{z}_{k_1})<f(\boldsymbol{y}),$$

which contradicts (14).

2) We assume $f(\boldsymbol{x})>f(\boldsymbol{y})$. Similarly, we have

$$f(\boldsymbol{y})<f(\boldsymbol{z})=f(\boldsymbol{x}), \tag{18}$$

and

$$f(\boldsymbol{z}+\mu\eta(\boldsymbol{x},\boldsymbol{z},\mu))=f(\boldsymbol{z})=f(\boldsymbol{x}),\ \forall\mu\in(0,1). \tag{19}$$

And then, define a sequence $\{\boldsymbol{z}_k\}$ by induction as follows:

$$\boldsymbol{z}_1=\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha),$$
$$\boldsymbol{z}_2=\boldsymbol{z}_1+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{z}_1,1-\alpha),$$
$$\vdots$$
$$\boldsymbol{z}_k=\boldsymbol{z}_{k-1}+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{z}_{k-1},1-\alpha),\ \forall k\geqslant 2,k\in\mathbf{N}.$$

According to the condition that, there exists $\alpha\in(0,1)$ such that, for all $\boldsymbol{x},\boldsymbol{y}\in K$ with $f(\boldsymbol{x})\neq f(\boldsymbol{y})$,

$$f(\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\},$$

and the assumption $f(\boldsymbol{y})<f(\boldsymbol{x})$, we have

$$f(\boldsymbol{z}_1)=f(\boldsymbol{y}+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha))<f(\boldsymbol{x}),$$
$$f(\boldsymbol{z}_2)=f(\boldsymbol{z}_1+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{z}_1,1-\alpha))<f(\boldsymbol{x}),$$
$$\vdots$$
$$f(\boldsymbol{z}_k)=f(\boldsymbol{z}_{k-1}+(1-\alpha)\eta(\boldsymbol{x},\boldsymbol{z}_{k-1},1-\alpha))<f(\boldsymbol{x}),\ \forall k\geqslant 2,k\in\mathbf{N}. \tag{20}$$

From Condition B2, it is easy to verify that $\boldsymbol{z}_k=\boldsymbol{y}+(1-\alpha^k)\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha^k),\ \forall k\in\mathbf{N}_+$. Since $0<\alpha<1$, then $\boldsymbol{z}_k\in K$ for all $k\in\mathbf{N}$. Choose a $k_1\in\mathbf{N}$ such that $1-\alpha^{k_1}>\lambda$ and denote $\beta=[(1-\alpha^{k_1})-\lambda]/(1-\lambda)$, then it is easy to know that $0<\beta<1$. Again from condition B_2

$$\boldsymbol{z}+\beta\eta(\boldsymbol{x},\boldsymbol{z},\beta)=\boldsymbol{y}+((1-\beta)\lambda+\beta)\eta(\boldsymbol{x},\boldsymbol{y},(1-\beta)\lambda+\beta)=\boldsymbol{y}+(1-\alpha^{k_1})\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha^{k_1})=\boldsymbol{z}_{k_1},$$

which, together with (20), yields $f(\boldsymbol{z}+\beta\eta(\boldsymbol{x},\boldsymbol{z},\beta))=f(\boldsymbol{z}_{k_1})<f(\boldsymbol{x})$, which contradicts (19). This completes the proof.

Remark 6 We delete the assumption $f(\boldsymbol{y}+\alpha\eta(\boldsymbol{x},\boldsymbol{y},\alpha))<\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$, while it's necessary in Theorem 10. Moreover, the proof is simplified.

Theorem 12 Let $f_i(\boldsymbol{x})$, $i=1,\cdots,m$, be semi-prequasi-invex with respect to $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ on K, with $\eta(\boldsymbol{x},\boldsymbol{y},\theta)$ satisfy Conditions B1, B2, and $\lim\limits_{\theta\downarrow 0}\theta\eta(\boldsymbol{x},\boldsymbol{y},\theta)=0$ for all $\boldsymbol{x},\boldsymbol{y}\in K$. And for each $i=1,\cdots,m$, there exists $\alpha_i\in(0,1)$ such that, for all $\boldsymbol{x},\boldsymbol{y}\in K$ with $f_i(\boldsymbol{x})\neq f_i(\boldsymbol{y})$,

$$f_i(\boldsymbol{y}+(1-\alpha_i)\eta(\boldsymbol{x},\boldsymbol{y},1-\alpha_i))<\max\{f_i(\boldsymbol{x}),f_i(\boldsymbol{y})\}.$$

Then, any local efficient solution of (MP) is the global efficient solution of (MP).

Proof Assume on the contrary that, there exists $\boldsymbol{y}\in K$ such that $\boldsymbol{y}$ is a local efficient solution of (MP), but is not a global efficient solution of (MP). Then, there exists $\boldsymbol{x}\in K$ such that

$$f_i(\boldsymbol{x})\leqslant f_i(\boldsymbol{y}),1\leqslant i\leqslant m, \tag{21}$$

and for some j,

$$f_j(\boldsymbol{x})<f_j(\boldsymbol{y}),1\leqslant j\leqslant m. \tag{22}$$

From the semi-prequasi-invexity of $f_1(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x})$, and (21), we have

$$f_i(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))\leqslant\max\{f_i(\boldsymbol{x}),f_i(\boldsymbol{y})\}=f_i(\boldsymbol{y}),\ \forall\lambda\in[0,1]. \tag{23}$$

Then, from Theorem 11, we know that $f_1(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x})$, is semistrictly semi-prequasi-invex functions.

And (22) implies that

$$f_j(\boldsymbol{y}+\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda))<\max\{f_j(\boldsymbol{x}),f_j(\boldsymbol{y})\}=f_j(\boldsymbol{y}),\ \forall\lambda\in(0,1). \tag{24}$$

The condition that $\lim\limits_{\lambda\downarrow 0}\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)=0$ together with (23) and (24), we get $\boldsymbol{y}$ is not a local efficient solution of (MP), which is a contradiction.

Theorem 13 Let $f_1(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x})$, be semistrictly semi-prequasi-invex function with respect to the same vector-valued function η, and $\lim\limits_{\lambda\downarrow 0}\lambda\eta(\boldsymbol{x},\boldsymbol{y},\lambda)=0$. Then, any local weakly efficient solution of (MP) is a global weakly efficient solution of (MP).

Theorem 14 Let $f_i(\boldsymbol{x})$, $i=1,\cdots,m$, be semi-prequasi-invex with respect to the vector-valued function η on K, with $\lim\limits_{\theta\downarrow 0}\theta\eta(\boldsymbol{x},\boldsymbol{y},\theta)=0$, and for some k, let $f_k(\boldsymbol{x})$ be strictly semi-prequasi-invex function with respect to the same vector-valued function η. Suppose that there exists $\lambda=(\lambda_1,\cdots,\lambda_m)\geqslant 0$, with $\lambda_k>0$, $k\in\{1,\cdots,m\}$, such that $\boldsymbol{y}\in K$ is a local solution of $\min\lambda^{\mathrm{T}}f(\boldsymbol{x})$, s. t. $\boldsymbol{x}\in K$. Then, $\boldsymbol{y}$ is also a global efficient solution of (MP).

Proof Assume on the contrary that, $\boldsymbol{y}\in K$ is not a global efficient solution of (MP), i. e., there exists some $\boldsymbol{x}\in K$, $f(\boldsymbol{x})\neq f(\boldsymbol{y})$, such that $f_i(\boldsymbol{x})\leqslant f_i(\boldsymbol{y})$, $1<i<m$.

Then, for any $\beta\in[0,1]$, from semi-prequasi-invexity of $f_i(\boldsymbol{x})$, we have

$$f_i(\boldsymbol{y}+\beta\eta(\boldsymbol{x},\boldsymbol{y},\beta))\leqslant\max\{f_i(\boldsymbol{x}),f_i(\boldsymbol{y})\}=f_i(\boldsymbol{y}),$$

also from the strictly prequasi-invexity of $f_k(\boldsymbol{x})$, we obtain

$$f_k(\boldsymbol{y}+\beta\eta(\boldsymbol{x},\boldsymbol{y},\beta))<\max\{f_k(\boldsymbol{x}),f_k(\boldsymbol{y})\}=f_k(\boldsymbol{y}),\ \forall\beta\in(0,1).$$

Hence, by $\lambda=(\lambda_1,\cdots,\lambda_m)\geqslant 0$, with $\lambda_k>0$, $k\in\{1,\cdots,m\}$, we have

$$\sum_{i=1}^{m}\lambda_i f_i(y+\beta\eta(\boldsymbol{x},\boldsymbol{y},\beta))<\sum_{i=1}^{m}\lambda_i f_i(\boldsymbol{y}),\ 0<\beta<1.$$

That is, $\lambda^{\mathrm{T}}f(\boldsymbol{y}+\beta\eta(\boldsymbol{x},\boldsymbol{y},\beta))<\lambda^{\mathrm{T}}f(\boldsymbol{y})$, $0<\beta<1$. Which is a contradiction.

Theorem 15 Let $f_i(\boldsymbol{x})$, $i=1,\cdots,m$, be semistrictly semi-prequasi-invex with respect to the vector-valued function η on K, and $\lim\limits_{\theta\downarrow 0}\theta\eta(\boldsymbol{x},\boldsymbol{y},\theta)=0$. Suppose there exists $\lambda=(\lambda_1,\cdots,\lambda_m)\geqslant 0$, with $\lambda_k>0$, $k\in\{1,\cdots,m\}$, such that $\boldsymbol{y}\in K$ is a local solution of $\min\lambda^{\mathrm{T}}f(\boldsymbol{x})$, s. t. $\boldsymbol{x}\in K$. Then, $\boldsymbol{y}$ is also a global weakly efficient solution of (MP).

Remark 7 If $m=1$ in (MP), then the mutiobjective mathematical programming (MP) reduces to a single objective mathematical programming (P_1).

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运筹学与控制论

半预拟不变凸性的性质与应用注记

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摘要:【目的】对半预拟不变凸性及其应用进行了深入研究。【方法】借助假设条件 B1, B2 和稠密性结果。【结果】首先，在更弱的假设下，获得了半预拟不变凸性的新刻画。然后，分别给出了无约束与不等式约束情形下半预拟不变凸型数学规划问题的最优性条件。最后，得到了半预拟不变凸性在多目标规划问题中的几个应用型结果，并举例说明所得结果。【结论】所得结果推广和改进了最近的一些文献。

关键词:性质;半连续性;半预拟不变凸函数;非线性规划;多目标规划问题

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