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# 一类不可微多目标规划的 Mond-Weir 型对偶*

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**摘要:**【目的】研究了一类不可微的多目标规划问题，其中目标函数包含支撑函数，约束包含等式和不等式。【方法】给出了该问题的一类 Mond-Weir 型对偶模型，利用 $G$-KKT 最优性必要条件和 $G$-不变凸性证明了原问题与对偶问题的对偶结果。【结果】在适当条件下，得到该问题与对偶问题的弱对偶定理、强对偶定理、逆对偶定理和非极大逆对偶定理，并进行了证明。【结论】将相关结论推广到了非可微情形。

**关键词:**不可微规划；多目标规划；Mond-Weir 型对偶；$G$-不变凸

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2007 年，Antczak 提出了一类新的广义凸性称为 $G$-不变凸[1]。2009 年，Antczak[2-3]研究了这一凸性条件下的一类多目标规划的最优性和对偶问题。随后，Kim 等人[4]将文献[2]的结论推广到非可微的情形。赵洁在文献[5]中研究了这一凸性条件下的一类非可微多目标规划的 Wolfe 对偶模型。受以上文献的启发[1-6]，本文对一类非可微多目标规划问题进行研究，其中目标包含支撑函数，约束条件包含等式和不等式，给出了该问题的 Mond-Weir 对偶模型及对偶结果。

## 1 预备知识

**定义 1**[2]　$f:X\to\mathbf{R}^k$ 定义在非空集 $X\subset\mathbf{R}^n$ 上可微向量值函数，$\boldsymbol{u}\in X$，若存在一个可微向量值函数 $G_f=(G_{f_1},\cdots,G_{f_k}):\mathbf{R}\to\mathbf{R}^k$，其中 $G_{f_i}:I_{f_i}(X)\to\mathbf{R}$ 是严格增函数，存在 $\eta:X\times X\to\mathbf{R}^n$，使得对于 $\forall\boldsymbol{x}\in X(\boldsymbol{x}\neq\boldsymbol{u})$，$\forall i=1,\cdots,k$ 有 $G_{f_i}(f_i(\boldsymbol{x}))-G_{f_i}(f_i(\boldsymbol{u}))-G'_{f_i}(f_i(\boldsymbol{u}))\nabla f_i(\boldsymbol{u})\eta(\boldsymbol{x},\boldsymbol{u})\geqq 0(>)$。则 $f$ 称为 $\boldsymbol{u}$ 关于 $\eta$ 的(严格)$G_f$-不变凸。

**定义 2**[7]　$C$ 为 $\mathbf{R}^n$ 的一个紧凸集，定义 $C$ 的支撑函数为 $s(\boldsymbol{x}|C):=\max\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}:\boldsymbol{y}\in C\}$。即存在 $\boldsymbol{z}\in\mathbf{R}^n$ 使 $s(\boldsymbol{y}|C)\geqslant s(\boldsymbol{x}|C)+\boldsymbol{z}^{\mathrm{T}}(\boldsymbol{y}-\boldsymbol{x})$，$\forall\boldsymbol{y}\in C$ 成立。$\boldsymbol{z}^{\mathrm{T}}\boldsymbol{x}=s(\boldsymbol{x}|C)$ 与之等价，其中 $s(\boldsymbol{x}|C)$ 的次微分为 $\partial s(\boldsymbol{x}|C)=\{\boldsymbol{z}\in C|\boldsymbol{z}^{\mathrm{T}}\boldsymbol{x}=s(\boldsymbol{x}|C)\}$。

**定义 3**　可行点 $\bar{\boldsymbol{x}}$ 称为问题(NMP)的弱 Pareto 解，若不存在 $\boldsymbol{x}\in D$ 使得

$$G_{f(\boldsymbol{x})+\boldsymbol{x}^{\mathrm{T}}\omega}f(\boldsymbol{x})+s(\boldsymbol{x}|C)<G_{f(\bar{\boldsymbol{x}})+\bar{\boldsymbol{x}}^{\mathrm{T}}\omega}f(\bar{\boldsymbol{x}})+s(\bar{\boldsymbol{x}}|C)。$$

**命题 1**　$\bar{z}$ 是多目标规划的可行解，令 $G_{f_i(\cdot)+(\cdot)^{\mathrm{T}}\omega_i}$，$i=1,2,\cdots,k$ 是 $I_{f_i(\cdot)+(\cdot)^{\mathrm{T}}\omega_i}(X)$ 上的连续实值严格增函数。定义：

$$W=\{G_{f_1(\cdot)+(\cdot)^{\mathrm{T}}\omega_1}(f_1(\boldsymbol{x})+s(\boldsymbol{x}|C_1)),\cdots,G_{f_k(\cdot)+(\cdot)^{\mathrm{T}}\omega_k}(f_k(\boldsymbol{x})+s(\boldsymbol{x}|C_k)):\boldsymbol{x}\in X\}\subset\mathbf{R}^k,$$

$$\bar{z}=(G_{f_1(\cdot)+(\cdot)^{\mathrm{T}}\omega_1}(f_1(\bar{\boldsymbol{x}})+s(\bar{\boldsymbol{x}}|C_1)),\cdots,G_{f_k(\cdot)+(\cdot)^{\mathrm{T}}\omega_k}(f_k(\bar{\boldsymbol{x}})+s(\bar{\boldsymbol{x}}|C_k)))\in W。$$

则多目标规划中 $\bar{\boldsymbol{x}}$ 是弱 Pareto 解当且仅当对应的 $\bar{z}$ 是 $W$ 中的弱 Pareto 解。

**定义 4**　称多目标规划在 $\bar{\boldsymbol{x}}\in D$ 满足 Kuhn-Tucker 约束规格，若

$$C(D,\bar{\boldsymbol{x}})=\{d\in\mathbf{R}^n:\nabla g_j(\bar{\boldsymbol{x}})d\leqq 0,j\in J(\bar{\boldsymbol{x}}),\nabla h_t(\bar{\boldsymbol{x}})d=0,t\in T\}。$$

本文研究如下的非可微多目标问题(NMP)[4]：

$$\min G_{F_1}((f_1(\boldsymbol{x})+s(\boldsymbol{x}|C_1)),\cdots,G_{F_k}(f_k(\boldsymbol{x})+s(\boldsymbol{x}|C_k))),$$

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$$\text{s.t.}\ (G_{g_1}(g_1(\boldsymbol{x})),\cdots,G_{g_m}(g_m(\boldsymbol{x})))\leqq 0,$$
$$(G_{h_1}(h_1(\boldsymbol{x})),\cdots,G_{h_p}(h_p(\boldsymbol{x})))=0。$$

其中,$f_i:X\to\mathbf{R},i\in I=\{1,2,\cdots,k\}$,$g_j:X\to\mathbf{R},j\in J=\{1,2,\cdots,m\}$,$h_t:X\to\mathbf{R},t\in T=\{1,2,\cdots,p\}$是非空开集$X\subset\mathbf{R}^n$上的可微函数,$G_{F_i},i\in I$,$G_{g_j},j\in J$,$G_{h_t},t\in T$是可微实值严格增函数,令$D=\{\boldsymbol{x}\in X:G_{g_j}(g_j(\boldsymbol{x}))\leqq 0,j\in J,G_{h_t}(h_t(\boldsymbol{x}))=0,t\in T\}$为问题(NMP)的可行解集,有$F_i=f_i(\cdot)+(\cdot)^{\mathrm{T}}\omega_i$。定义不等式约束$\boldsymbol{z}\in D$上的起作用集为$J(\boldsymbol{z}):=\{j\in J:G_{g_j}(g_j(\boldsymbol{z}))=0\}$,目标函数的指标集为$I(\boldsymbol{z}):=\{i\in I:\lambda_i>0\}$,对应的Lagrange乘子不为零。

## 2 结论及证明

**定理** 1($G$-KKT必要条件)[5] 设$G_{F_i},i\in I$是$I_{F_i}(D)$上的可微实值严格递增函数,$G_{g_j},j\in J$是$I_{g_j}(D)$上的可微实值严格递增函数,$G_{h_t},t\in T$是$I_{h_t}(D)$上的可微实值严格递增函数,$G_{h_t},t\in T$线性无关,$F_i=f_i(\cdot)+(\cdot)^{\mathrm{T}}\omega_i$。假设存在问题$\boldsymbol{z}^*\in\mathbf{R}^n$,使$\langle G'_{g_j}(g_j(\overline{\boldsymbol{x}}))\nabla g_j(\overline{\boldsymbol{x}}),\boldsymbol{z}^*\rangle<0,j\in J(\overline{\boldsymbol{x}})$,$\langle G'_{h_t}(h_t(\overline{\boldsymbol{x}}))\nabla h_t(\overline{\boldsymbol{x}}),\boldsymbol{z}^*\rangle=0$,$t=1,2,\cdots,p$。若$\overline{\boldsymbol{x}}\in D$是问题(NMP)的一个弱Pareto最优点,则存在$\lambda\in\mathbf{R}_+^k,\xi\in\mathbf{R}_+^m,\mu\in\mathbf{R}^p,\omega_i\in C_i,\ i=1,2,\cdots,k$,使得

$$\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(\overline{\boldsymbol{x}})+\overline{\boldsymbol{x}}^{\mathrm{T}}\omega_i)(\nabla f_i(\overline{\boldsymbol{x}})+\omega_i)+\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(\overline{\boldsymbol{x}}))\nabla g_j(\overline{\boldsymbol{x}})+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(\overline{\boldsymbol{x}}))\nabla h_t(\overline{\boldsymbol{x}})=0,\quad(1)$$

$$\xi_j G_{g_j}(g_j(\overline{\boldsymbol{x}}))=0,j\in J,\quad(2)$$

$$\langle\omega_i,\overline{\boldsymbol{x}}\rangle=s(\overline{\boldsymbol{x}}|C_i),i=1,2,\cdots,k,\quad(3)$$

$$\lambda\geqslant 0,\sum_{i=1}^{k}\lambda_i=1,\xi\geqq 0。\quad(4)$$

给出问题(NMP)的Mond-Weir对偶模型(NMWD)为

$$f(\boldsymbol{y})=(G_{F_1}(f_1+s(\boldsymbol{y}|C_1)),\cdots,G_{F_k}(f_k+s(\boldsymbol{y}|C_k)))\to\max$$
$$\Big[\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(\boldsymbol{y})+s(\boldsymbol{y}\mid C_i))(\nabla f_i(\boldsymbol{y})+\omega_i)+\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(\boldsymbol{y}))\nabla g_j(\boldsymbol{y})+$$
$$\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(\boldsymbol{y}))\nabla h_t(\boldsymbol{y})\Big]\cdot\eta(\boldsymbol{x},\boldsymbol{y})\geqq 0,\forall\boldsymbol{x}\in D,$$
$$\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(\boldsymbol{y}))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(\boldsymbol{y}))\geqq 0,\boldsymbol{y}\in X,$$
$$\lambda\in\mathbf{R}^n,\lambda\geqslant 0,\lambda\boldsymbol{e}=1,\xi\in\mathbf{R}^m,\xi\geqq 0,\mu\in\mathbf{R}^p。$$

**引理** 1[3] 令$(\boldsymbol{y},\lambda,\xi,\mu)$为问题(NMWD)的任意一个可行点。若$g$在$\boldsymbol{y}\in pr_X W_1$上是关于$\eta$的$G_g$-不变凸函数,$h_t,t\in T^+(\boldsymbol{y})$在$\boldsymbol{y}\in pr_X W_1$上是关于$\eta$的$G_{h_t}$-不变凸函数,$h_t,t\in T^-(\boldsymbol{y})$在$\boldsymbol{y}\in pr_X W_1$上是关于$\eta$的$G_{h_t}$-不变凹函数,$G_{g_j}(0)=0,j\in J$,$G_{h_t}(0)=0,t\in T^+(\boldsymbol{y})\cup T^-(\boldsymbol{y})$,则

$$\Big[\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(\boldsymbol{y}))\nabla g_j(\boldsymbol{y})+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(\boldsymbol{y}))\nabla h_t(\boldsymbol{y})\Big]\cdot\eta(\boldsymbol{x},\boldsymbol{y})\leqq 0,\forall\boldsymbol{x}\in D。\quad(5)$$

**定理** 2(弱对偶) $\boldsymbol{x}$和$(\boldsymbol{y},\lambda,\xi,\mu)$分别是问题(NMP)和(NMWD)的任意可行解。进一步假设$F$在$\boldsymbol{y}\in pr_X W_1$是关于$\eta$的$G_F$-不变凸函数,$g$在$\boldsymbol{y}\in pr_X W_1$是关于$\eta$的$G_g$-不变凸函数,$h_t,t\in T^+(\boldsymbol{y})$在$\boldsymbol{y}\in pr_X W_1$是关于$\eta$的$G_{h_t}$-不变凸函数,$h_t,t\in T^-(\boldsymbol{y})$在$\boldsymbol{y}\in pr_X W_1$上是关于$\eta$的$G_{h_t}$-不变凹函数,$G_{g_j}(0)=0,j\in J$,$G_{h_t}(0)=0,t\in T^+(\boldsymbol{y})\cup T^-(\boldsymbol{y})$,则$f(\boldsymbol{x})+s(\boldsymbol{x}|C)\not<f(\boldsymbol{y})+s(\boldsymbol{y}|C)$。

**证明** 反证法。假设$f(\boldsymbol{x})+s(\boldsymbol{x}|C)<f(\boldsymbol{y})+s(\boldsymbol{y}|C)$,则有

$$f_i(\boldsymbol{x})+s(\boldsymbol{x}|C_i)<f_i(\boldsymbol{y})+s(\boldsymbol{y}|C_i),i\in I。\quad(6)$$

根据定义,$G_{F_i},i\in I$是$D\cup pr_X W_1$上的严格增函数,由(7)式得

$$G_{F_i}(f_i(\boldsymbol{x})+s(\boldsymbol{x}|C_i))<G_{F_i}(f_i(\boldsymbol{y})+s(\boldsymbol{y}|C_i)),i\in I。\quad(7)$$

由假设,$F$在$\boldsymbol{y}\in pr_X W_1$关于$\eta$是$G_F$-不变凸,则

$$G_{F_i}(f_i(\boldsymbol{x})+s(\boldsymbol{x}|C_i))-G_{F_i}(f_i(\boldsymbol{y})+s(\boldsymbol{y}|C_i))-G'_{F_i}(f_i(\boldsymbol{y})+s(\boldsymbol{y}|C_i))(\nabla f_i(\boldsymbol{y})+\omega_i)\eta(\boldsymbol{x},\boldsymbol{y})\geqq 0,i\in I。\quad(8)$$

因此,由(7),(8)式得$G'_{F_i}(f_i(\boldsymbol{y})+s(\boldsymbol{y}|C_i))(\nabla f_i(\boldsymbol{y})+\omega_i)\eta(\boldsymbol{x},\boldsymbol{y})<0,i\in I$。

又因为$(\boldsymbol{y},\lambda,\xi,\mu)$是问题(NMWD)的可行解，由问题(NMWD)的约束条件

$$\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(\boldsymbol{y})+s(\boldsymbol{y}\mid C_i))(\nabla f_i(\boldsymbol{y})+\omega_i)\eta(\boldsymbol{x},\boldsymbol{y})<0,i\in I。\tag{9}$$

由假设 $g$ 在 $\boldsymbol{y}\in pr_X W_1$ 关于 $\eta$ 是 $G_g$-不变凸，$h_t,t\in T^+(\boldsymbol{y})$在 $\boldsymbol{y}\in pr_X W_1$ 关于 $\eta$ 是 $G_{h_t}$-不变凸，$h_t,t\in T^-(\boldsymbol{y})$在 $\boldsymbol{y}\in pr_X W_1$ 关于 $\eta$ 是 $G_{h_t}$-不变凹，$G_{g_j}(0)=0,j\in J,G_{h_t}(0)=0,t\in T^+(\boldsymbol{y})\cup T^-(\boldsymbol{y})$，则根据引理 1，把(5)，(9)式左右两边分别相加，有

$$\Big[\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(\boldsymbol{y})+s(\boldsymbol{y}\mid C_i))(\nabla f_i(\boldsymbol{y})+\omega_i)+\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(\boldsymbol{y}))\nabla g_j(\boldsymbol{y})+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(\boldsymbol{y}))\nabla h_t(\boldsymbol{y})\Big]\eta(\boldsymbol{x},\boldsymbol{y})<0$$

与问题(NMWD)的约束条件矛盾。　　证毕

**定理** 3(强对偶)　假设$\bar{\boldsymbol{x}}\in D$ 是问题(NMP)的一个(弱)Pareto 解，$\bar{\boldsymbol{x}}$满足 Kuhn-Tucker 约束规格，$G_{h_t}(0)=0$，$t\in T^+(\boldsymbol{y})\cup T^-(\boldsymbol{y})$。则存在$\bar{\lambda}\in\mathbf{R}_+^k,\bar{\xi}\in\mathbf{R}_+^m,\bar{\mu}\in\mathbf{R}^p,(\bar{\lambda}\geqslant 0,\bar{\xi}\geqq 0),\bar{\lambda}>0,\bar{\xi}\geqq 0$，使得$(\bar{\boldsymbol{x}},\bar{\lambda},\bar{\xi},\bar{\mu})$是问题(NMWD)的可行解。若定理 2 成立，则$(\bar{\boldsymbol{x}},\bar{\lambda},\bar{\xi},\bar{\mu})$是问题(NMWD)的(弱)Pareto 解，且问题(NMP)和(NMWD)目标函数值相等。

**证明**　$\bar{\boldsymbol{x}}$是问题(NMP)的(弱)Pareto 解，则存在$\bar{\lambda}\in\mathbf{R}_+^k,\bar{\xi}\in\mathbf{R}_+^m,\bar{\mu}\in\mathbf{R}^p,(\bar{\lambda}\geqslant 0,\bar{\xi}\geqq 0),\bar{\lambda}>0,\bar{\xi}\geqq 0$，使得 $G$-KKT 必要条件(1)～(4)式成立，即有

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{\boldsymbol{x}}))=0,j\in J。\tag{10}$$

由假设 $G_{h_t}(0)=0,t\in T^+(\boldsymbol{y})\cup T^-(\boldsymbol{y}),\bar{\boldsymbol{x}}\in D$，有

$$\sum_{t=1}^{p}\bar{\mu}_t G_{g_t}(h_t(\bar{\boldsymbol{x}}))\geqq 0。\tag{11}$$

综合 $G$-KKT 必要条件(1)，(10)和(11)式可得$(\bar{\boldsymbol{x}},\bar{\lambda},\bar{\xi},\bar{\mu})$在问题(NMWD)中的可行性。同时，由弱对偶定理可得$(\bar{\boldsymbol{x}},\bar{\lambda},\bar{\xi},\bar{\mu})$是问题(NMWD)的(弱)Pareto 解。　　证毕

**定理** 4(逆对偶)　$(\bar{\boldsymbol{y}},\bar{\lambda},\bar{\xi},\bar{\mu})$是问题(NMWD)的弱 Pareto 解(Pareto 解)且$\bar{\boldsymbol{y}}\in D$，假设 $F$ 是上 $\boldsymbol{y}\in pr_X W_1$ 关于$\eta$ 的$G_F$-不变凸函数，$g$ 是 $\boldsymbol{y}\in pr_X W_1$ 上关于 $\eta$ 的 $G_g$-不变凸函数，$h_t,t\in T^+(\boldsymbol{y})$是 $\boldsymbol{y}\in pr_X W_1$ 上关于 $\eta$ 的 $G_{h_t}$-不变凸函数，$h_t,t\in T^-(\boldsymbol{y})$是 $\boldsymbol{y}\in pr_X W_1$ 上关于 $\eta$ 的 $G_{h_t}$-不变凹函数，则$\bar{\boldsymbol{y}}$是问题(NMP)的弱 Pareto 解(Pareto 解)。

**证明**　反证法。设$\bar{\boldsymbol{y}}$不是问题(NMP)的弱 Pareto 解，即存在$\tilde{\boldsymbol{x}}\in D$ 使 $f(\tilde{\boldsymbol{x}})+s(\tilde{\boldsymbol{x}}\mid C)<f(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}\mid C)$。

由假设 $F$ 在$\bar{\boldsymbol{y}}$关于 $\eta$ 是 $G_F$-不变凸函数，$G_{F_i},i\in I$ 是 $D\cup pr_X W_1$ 上的严格增函数，$\bar{\lambda}\geqslant 0$，有

$$\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\tilde{\boldsymbol{x}})+s(\tilde{\boldsymbol{x}}\mid C_i))<\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}\mid C_i)),i\in I。\tag{12}$$

因为$(\bar{\boldsymbol{y}},\bar{\lambda},\bar{\xi},\bar{\mu})$是问题(NMWD)的弱 Pareto 解，则由 $G$-KKT 必要条件(2)式，有

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{\boldsymbol{y}}))=0,j\in J。\tag{13}$$

由$\tilde{\boldsymbol{x}}\in D$，有

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{\boldsymbol{x}}))\leqslant 0,j\in J。\tag{14}$$

联立(13)，(14)式，有

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{\boldsymbol{x}}))\leqq\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{\boldsymbol{y}})),j\in J。\tag{15}$$

由$\bar{\boldsymbol{y}}\in D$ 和$\tilde{\boldsymbol{x}}\in D$，有

$$\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{\boldsymbol{x}}))-\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{\boldsymbol{y}}))=0。\tag{16}$$

因为 $F$ 是 $\boldsymbol{y}\in pr_X W_1$ 上关于 $\eta$ 的 $G_F$-不变凸函数，$g$ 是 $\boldsymbol{y}\in pr_X W_1$ 上关于 $\eta$ 的 $G_g$-不变凸函数，$h_t,t\in$

$T^{+}(\boldsymbol{y})$是$\boldsymbol{y}\in pr_X W_1$上关于$\eta$的$G_{h_t}$-不变凸函数,$h_t,t\in T^{-}(\boldsymbol{y})$是$\boldsymbol{y}\in pr_X W_1$上关于$\eta$的$G_{h_t}$-不变凹函数,则有如下式子成立:

$$G_{F_i}(f_i(\tilde{\boldsymbol{x}})+s(\tilde{\boldsymbol{x}}|C_i))-G_{F_i}(f_i(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}|C_i))-G'_{F_i}(f_i(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}|C_i))(\nabla f_i(\bar{\boldsymbol{y}})+\omega_i)\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})\geqq 0,i\in I,$$

$$G_{g_j}(g_j(\tilde{\boldsymbol{x}}))-G_{g_j}(g_j(\bar{\boldsymbol{y}}))-G'_{g_j}(g_j(\bar{\boldsymbol{y}}))\nabla g_j(\bar{\boldsymbol{y}})\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})\geqq 0,j\in J,$$

$$G_{h_t}(h_t(\tilde{\boldsymbol{x}}))-G_{h_t}(h_t(\bar{\boldsymbol{y}}))-G'_{h_t}(h_t(\bar{\boldsymbol{y}}))\nabla h_t(\bar{\boldsymbol{y}})\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})\geqq 0,t\in T^{+}(\bar{\boldsymbol{y}}),$$

$$G_{h_t}(h_t(\tilde{\boldsymbol{x}}))-G_{h_t}(h_t(\bar{\boldsymbol{y}}))-G'_{h_t}(h_t(\bar{\boldsymbol{y}}))\nabla h_t(\bar{\boldsymbol{y}})\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})\leqq 0,t\in T^{-}(\bar{\boldsymbol{y}})。$$

由$(\bar{\boldsymbol{y}},\bar{\lambda},\bar{\xi},\bar{\mu})$在问题(NMWD)的可行性,有

$$\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\tilde{\boldsymbol{x}})+s(\tilde{\boldsymbol{x}}\mid C_i))-\sum_{i=1}^{k}\bar{\lambda}_i G_{F_i}(f_i(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}\mid C_i))\geqq$$
$$\sum_{i=1}^{k}\bar{\lambda}_i G'_{F_i}(f_i(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}\mid C_i))(\nabla f_i(\bar{\boldsymbol{y}})+\omega_i)\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}}),i\in I,\tag{17}$$

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{\boldsymbol{x}}))-\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{\boldsymbol{y}}))\geqq\sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{\boldsymbol{y}}))\nabla g_j(\bar{\boldsymbol{y}})\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}}),j\in J,\tag{18}$$

$$\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{\boldsymbol{x}}))-\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{\boldsymbol{y}}))\geqq\sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{\boldsymbol{y}}))\nabla h_t(\bar{\boldsymbol{y}})\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}}),t\in T(\bar{\boldsymbol{y}})。\tag{19}$$

由(12),(15)~(16),(17)~(19)式,得

$$\sum_{i=1}^{k}\bar{\lambda}_i G'_{F_i}(f_i(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}\mid C_i))(\nabla f_i(\bar{\boldsymbol{y}})+\omega_i)\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})<0,i\in I,\tag{20}$$

$$\sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{\boldsymbol{y}}))\nabla g_j(\bar{\boldsymbol{y}})\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})\leqq 0,j\in J,\tag{21}$$

$$\sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{\boldsymbol{y}}))\nabla h_t(\bar{\boldsymbol{y}})\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})\leqq 0,t\in T(\bar{\boldsymbol{y}})。\tag{22}$$

把(25)~(27)式左右两边分别相加,得

$$\Big[\sum_{i=1}^{k}\bar{\lambda}_i G'_{F_i}(f_i(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}|C_i))(\nabla f_i(\bar{\boldsymbol{y}})+\omega_i)+\sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{\boldsymbol{y}}))\nabla g_j(\bar{\boldsymbol{y}})+$$
$$\sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{\boldsymbol{y}}))\nabla h_t(\bar{\boldsymbol{y}})\Big]\cdot\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})<0,\tag{23}$$

与$(\bar{\boldsymbol{y}},\bar{\lambda},\bar{\xi},\bar{\mu})$在问题(NMWD)的可行性矛盾,得证。 证毕

**定理** 5(非极大逆对偶) $(\bar{\boldsymbol{y}},\bar{\lambda},\bar{\xi},\bar{\mu})$是问题(NMWD)的可行解且$\bar{\boldsymbol{y}}\in D$,假设$F$是$\boldsymbol{y}\in pr_X W_1$上关于$\eta$的(严格)$G_F$-不变凸函数,$g$是$\boldsymbol{y}\in pr_X W_1$上关于$\eta$的$G_g$-不变凸函数,$h_t,t\in T^{+}(\boldsymbol{y})$是$\boldsymbol{y}\in pr_X W_1$上关于$\eta$的$G_{h_t}$-不变凸函数,$h_t,t\in T^{-}(\boldsymbol{y})$是$\boldsymbol{y}\in pr_X W_1$上关于$\eta$的$G_{h_t}$-不变凹函数,$G_{g_j}(0)=0,j\in J,G_{h_t}(0)=0,t\in T^{+}(\boldsymbol{y})\cup T^{-}(\boldsymbol{y})$,则$\bar{\boldsymbol{y}}$是问题(NMP)的弱Pareto解(Pareto解)。

**证明** 反证法。设$\bar{\boldsymbol{y}}$不是问题(NMP)的弱Pareto解,即存在$\tilde{\boldsymbol{x}}\in D$使$f_i(\tilde{\boldsymbol{x}})+s(\tilde{\boldsymbol{x}}|C_i)<f_i(\bar{\boldsymbol{y}})+s(\bar{\boldsymbol{y}}|C_i)$成立,与定理4的证明相似,(20)式满足。

由在$F,g,h$上的假设和$(\bar{\boldsymbol{y}},\bar{\lambda},\bar{\xi},\bar{\mu})$是问题(NMWD)的可行解,(22)~(24)式可得。

将(18),(19)式左右两边分别相加,得:

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{\boldsymbol{x}}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{\boldsymbol{x}}))-\Big[\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{\boldsymbol{y}}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{\boldsymbol{y}}))\Big]\geqq$$
$$\Big[\sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{\boldsymbol{y}}))\nabla g_j(\bar{\boldsymbol{y}})+\sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{\boldsymbol{y}}))\nabla h_t(\bar{\boldsymbol{y}})\Big]\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}}),j\in J。\tag{24}$$

因为$G_{g_j},G_{h_t}$是严格增函数,$G_{g_j}(0)=0,j\in J,G_{h_t}(0)=0,t\in T^{+}(\boldsymbol{y})\cup T^{-}(\boldsymbol{y})$,所以

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{\boldsymbol{x}}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{\boldsymbol{x}}))\leqq 0。\tag{25}$$

因此由$(\bar{\boldsymbol{y}},\bar{\lambda},\bar{\xi},\bar{\mu})$在问题(NMWD)的可行性和(25)式,得:

$$\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\tilde{\boldsymbol{x}}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\tilde{\boldsymbol{x}}))-\Big[\sum_{j=1}^{m}\bar{\xi}_j G_{g_j}(g_j(\bar{\boldsymbol{y}}))+\sum_{t=1}^{p}\bar{\mu}_t G_{h_t}(h_t(\bar{\boldsymbol{y}}))\Big]\leqq 0。\tag{26}$$

由(24),(26)式,有

$$\left[\sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{\boldsymbol{y}}))\nabla g_j(\bar{\boldsymbol{y}})+\sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{\boldsymbol{y}}))\nabla h_t(\bar{\boldsymbol{y}})\right]\eta(\tilde{\boldsymbol{x}},\bar{\boldsymbol{y}})\leqq 0。\tag{27}$$

将(20),(27)式两边分别相加得到(23)式,矛盾,得证。 证毕

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**Operations Research and Cybernetics**

## Mond-Weir Duality for a Class of Nondifferentiable Multiobjective Programming


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**Abstract**: [**Purposes**] Mond-Weir type dual problem of a class of nondifferentiable multiobjective programs were studied, the problems with both inequality and equality constrains in which every component of the objective function contains a term involving the support function of a compact convex set were considered. [**Methods**] Mode-Weir type dual problem was formulated. $G$-KKT necessary optimality conditions and $G$-invex assumption were used to establish duality theorems relating the problem and the dual problems. [**Findings**] Weak duality theorems, strong duality theorem, converse duality theorem and no-maximal converse duality theorem were established under suitable conditions. [**Conclusions**] The work generalized some related results to the nondifferentiable case.



**Keywords**: nondifferentiable programming; multiobjective programming; Mond-Weir duality; $G$-invex function


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