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Algebraic Characterizations of Weakly E-efficient Solutions for Vector Optimization Problems*

LIN An[1], ZHU Qiao[2], ZHAO Kequan[2]

(1. Mathematics and Computer College, Chongqing Normal University Foreign Trade and Business College, Chongqing 401520;
2. College of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China)

Abstract: [**Purposes**] A class of vector optimization problems with set-valued maps are studied. [**Methods**] By means of the basic notion of algebraic interior, an alternative theorem with nearly E-subconvexlikeness of set-valued maps is established via improvement set and as applications, a class of vector optimization problems with set-valued maps are further considered. [**Findings**] Linear scalarization theorem and Lagrange multiplier theorem are given for weakly E-efficient solutions via improvement set and algebraic interior. Some examples also are presented to illustrate the main results. [**Conclusions**] The main results improve and generalize some corresponding results in the literatures.

Keywords: vector optimization problems with set-valued maps; improvement set; algebraic interior; nearly E-subconvexlikeness; weakly E-efficient solutions; scalarization

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In recent years, study on the theory of vector optimization has been paid more attentions. So far as there are a lot of fundamental and important research results[1-3]. In particular, some scholars obtained some characterizations of various kinds of solutions for vector optimization problems under some suitable generalized convexity[4-6].

Approximate solutions have been playing an important role when there are no exact solutions for a class of vector optimization problems. Loridan[7] initially introduced the concept of ε-efficient solutions. Rong and Wu[8] introduced the concept of weakly ε-efficient solutions and obtained some characterizations such as linear scalarization theorem and Lagrange multiplier theorem.

With the development of various kinds of exact and approximate solutions, it becomes one meaningful research subject that how to propose some unified solution concepts and obtain some new characterizations in a unified framework for vector optimization problems. Chicco et al.[9] proposed the concepts of improvement set and E-efficient solutions via improvement set and obtained some characterizations. E-efficiency includes some known exact and approximate solutions as its special cases. Improvement set is close related to the free disposal set proposed by Debreu[10]. As a kind of important tool, improvement set has been used extensively in vector

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The first author biography: LIN An, female, research fields is vector optimization theory and method; E-mail: mathalin@163.com; **Corresponding author**: ZHAO Kequan, professor, E-mail: kequanz@163.com

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第一作者简介:林安,女,研究方向为向量优化理论与方法,E-mail: mathalin@163.com;**通信作者**:赵克全,教授,E-mail:kequanz@163.com

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optimization[11-15]. Especially, Zhao et al.[13] introduced the concept of weakly E-efficient solutions and established some linear scalarization theorem and Lagrange multiplier theorem under the nearly E-subconvexlikeness and the corresponding alternative theorem.

It is worth noting that many solution concepts depend on the nonemptiness of the ordering cone in the image space of a vector optimization problem. Hence, it is one meaningful and valuable research topic how to introduce some new solution concepts and give some characterizations by means of some generalized interior tools. Some classical tools generalized interiors such as algebraic interior, relatively topological interior and relatively algebraic interior[16-18]. Some scholars also have proposed some new solution concepts via generalized interiors of the ordering cone and obtained some characterizations[19-22].

Motivated by the works of [5, 9, 13, 18], this article focus on establishing an alternative theorem with nearly E-subconvexlikeness of set-valued maps via improvement set and algebraic interior in a general real linear space and giving the linear scalarization theorem and Lagrange multiplier theorem for weakly E-efficient solutions via improvement set and algebraic interior. It also presents some examples to illustrate the main assumption conditions and results. The related research works improve and generalize some known results in the literatures.

1 Preliminaries

Let X, Y and Z be three real linear spaces, Y^* and Z^* be the linear dual space of Y and Z respectively, $\mathbf{R}^n$ be the n-dimensional Euclidean space, $\mathbf{R}^n_+$ be the nonnegative orthant, $\mathbf{R}^n_{++}$ be the positive orthant. For a nonempty subset A in Y, the algebraic interior and vector closure of A denoted by $\mathrm{cor}A$ and $\mathrm{vcl}A$ are respectively defined as

$$\mathrm{cor}A=\{y\in Y \mid \forall h\in Y,\ \exists\varepsilon>0,\forall t\in[0,\varepsilon], y+th\in A\},$$
$$\mathrm{vcl}A=\{y\in Y \mid \exists h\in Y,\ \forall\varepsilon>0,\exists t\in(0,\varepsilon], y+th\in A\}.$$

A is said to be proper if $A\neq\varnothing$ and $A\neq Y$. Moreover, the cone hull and the positive dual cone of A are respectively defined as $\mathrm{cone}A=\{\alpha a \mid \forall\alpha\geqslant 0, \forall a\in A\}$; $A^+=\{\mu\in Y^* \mid \langle\mu,y\rangle\geqslant 0, \forall y\in A\}$.

Let K be a proper convex cone in Y. A is said to be a free disposal set with respect to K if $A+K=A$.

Definition 1[9, 12-14] Let $E\subset Y$ and K be a proper convex cone. If $0\notin E$ and E is a free disposal set with respect to K, then E is said to be an improvement set with respect to K.

Lemma 1[3] Let $A\subset Y$ and $B\subset Y$ be two nonempty convex sets. If $\mathrm{cor}A\neq\varnothing$ and $\mathrm{cor}A\cap B=\varnothing$, then there exist $\mu\in Y^*\setminus\{0_{Y^*}\}$ and $\alpha\in\mathbf{R}$ such that $\langle\mu,a\rangle\leqslant\alpha\leqslant\langle\mu,b\rangle$, $\forall a\in A$, $\forall b\in B$ and $\langle\mu,a\rangle<\alpha$, $\forall a\in\mathrm{cor}A$.

Lemma 2[19] Let $K\subset Y$ be a proper convex cone with nonempty algebraic interior. If $k\in\mathrm{cor}K$ and $\mu\in K^+\setminus\{0_{Y^*}\}$, then $\langle\mu,k\rangle>0$.

2 Alternative theorem with nearly E-subconvexlikeness

In this section introduces the concept of nearly E-subconvexlikeness of set-valued maps by using algebraic interior, vector closure and improvement set in a real linear space. Furthermore, this paper will establish the alternative theorem with nearly E-subconvexlikeness.

Lemma 3 Let $K\subset Y$ be a proper convex cone with nonempty algebraic interior. If $E\subset Y$ is a free disposal set with respect to K, then $\mathrm{cor}E=E+\mathrm{cor}K$.

Proof Let $y\in E+\mathrm{cor}K$, then there exist $e\in E$ and $k\in\mathrm{cor}K$ such that $y=e+k$. From the definition of algebraic interior, it can obtain that for any $h\in Y$, there exists $\varepsilon>0$ such that $k+th\in K$ for all $t\in[0,\varepsilon]$. Therefore, $y+th=e+k+th\in E+K=E$. Hence $y\in\mathrm{cor}E$ and it follows that $\mathrm{cor}E\neq\varnothing$.

Conversely, let $e\in\mathrm{cor}E$ and $k\in\mathrm{cor}K$, then from the definition of algebraic interior, there exists $\varepsilon>0$ such that $e-\varepsilon k\in E$. Therefore, $e\in E+\varepsilon k\subset E+\mathrm{cor}K$.

Remark 1 From Lemma 3, it is clear that $\mathrm{cor}K\neq\varnothing$ implies $\mathrm{cor}E\neq\varnothing$.

Lemma 4 Let $K\subset Y$ be a proper convex cone with nonempty topological interior. If $E\subset Y$ is a free disposal set with respect to K, then $\mathrm{vcl}(\mathrm{cone}E)=\mathrm{cl}(\mathrm{cone}E)$.

Proof　In fact, it is clear that $\mathrm{vcl}(\mathrm{cone}E) \subset \mathrm{cl}(\mathrm{cone}E)$. In the following, it only need to prove

$$\mathrm{cl}(\mathrm{cone}E) \subset \mathrm{vcl}(\mathrm{cone}E). \tag{1}$$

Let $e \in \mathrm{cl}(\mathrm{cone}E)$ and $h \in \mathrm{int}K$. For any given $\varepsilon > 0$, taking $t = \varepsilon$, then $th \in \mathrm{int}K$. Therefore,

$$e + th \in \mathrm{cl}(\mathrm{cone}E) + \mathrm{int}K = \mathrm{cl}(\mathrm{cone}E \backslash \{0\}) + \mathrm{int}K = \mathrm{cone}E \backslash \{0\} + \mathrm{int}K.$$

Hence there exist $\lambda_1, \lambda_2 > 0$, $e' \in E$ and $k \in \mathrm{int}K$ such that

$$e + th = \lambda_1 e' + \lambda_2 k = \lambda_1 \left(e' + \frac{\lambda_2}{\lambda_1} k \right) \in \lambda_1 (e' + \mathrm{int}K).$$

Therefore, from E is a free disposal set with respect to K, there have $e + th \in \mathrm{cone}E$, which implies $e \in \mathrm{vcl}(\mathrm{cone}E)$ and so (1) does hold.

Remark 2　If E is not a free disposal set with respect to K in Lemma 4, then the conclusion is not necessarily valid. The following example illustrates it.

Example 1　Let $Y = \mathbf{R}^3$, $K = \mathbf{R}^3_+$ and $E = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid 1 \geqslant x_1 > 0, x_2 = x_1^2, x_3 = 1\}$. Clearly, K is a proper convex cone with nonempty topological interior, but E is not a free disposal set with respect to K. Moreover, $\mathrm{cone}E = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_3 \geqslant x_1 > 0, x_2 x_3 = x_1^2\} \cup \{0\}$. Take $y_0 = (0, 0, 1)$. It can verify that $y_0 \in \mathrm{cl}(\mathrm{cone}E)$ and $y_0 \notin \mathrm{vcl}(\mathrm{cone}E)$.

Remark 3　If $\mathrm{int}K = \varnothing$ in Lemma 4, then the conclusion also is not necessarily valid. The following example illustrates it.

Example 2　Let $Y = \mathbf{R}^4$, $K = \{(y_1, y_2, y_3, y_4) \in \mathbf{R}^4 \mid y_1 = y_2 = y_3 = 0, y_4 \geqslant 0\}$ and

$$E = \{(y_1, y_2, y_3, y_4) \in R^4 \mid 1 \geqslant y_1 > 0, y_2 = y_1^2, y_3 = 1, y_4 \geqslant 1\}.$$

Clearly, K is a proper convex cone and E is a free disposal set with respect to K, but $\mathrm{int}K = \varnothing$. Moreover,

$$\mathrm{cone}E = \{(y_1, y_2, y_3, y_4) \in \mathbf{R}^4 \mid y_4 \geqslant y_3 \geqslant y_1 > 0, y_2 y_3 = y_1^2\} \cup \{0\}.$$

Take $y_0 = (0, 0, 1, 0)$. It can verify that $y_0 \in \mathrm{cl}(\mathrm{cone}E)$ and $y_0 \notin \mathrm{vcl}(\mathrm{cone}E)$.

Remark 4　It is well-known that $\mathrm{vcl}E = \mathrm{cl}E$ under the conditions that E is a convex set and $\mathrm{int}E \neq \varnothing$. From the proof of Lemma 4, it observes that $\mathrm{vcl}E = \mathrm{cl}E$ also holds under the assumption conditions of Lemma 4.

Motivated by the idea of nearly E-subconvexlikeness in a real separated locally convex topological linear space proposed by Zhao et al.[13], the paper propose the following notion of nearly E-subconvexlikeness via algebraic interior and improvement set in a real linear space.

In the following, unless particularly stated, the paper assumes that $K \subset Y$ be a proper convex cone with nonempty algebraic interior and $E \subset Y$ be an improvement set with respect to K.

Definition 2　Let $S \subset X$ be a nonempty set and $F: S \to 2^Y$ be a set-valued map. F is said to be nearly E-subconvexlike on S, if $\mathrm{vcl}(\mathrm{cone}(F(S) + E))$ is a convex set.

Remark 5　If Y is a real separated locally convex topological linear space and $K \subset Y$ is a pointed closed convex cone with nonempty topological interior, then Definition 1 coincides with Definition 3.1 [13]. In fact, it only need to verify that

$$\mathrm{vcl}(\mathrm{cone}(F(S) + E)) = \mathrm{cl}(\mathrm{cone}(F(S) + E)). \tag{2}$$

Since E is an improvement set with respect to K, then $F(S) + E + K = F(S) + E$, i.e., $F(S) + E$ is a free disposal set with respect to K. Then it follows from Lemma 4 that (2) does hold.

For a nonempty set A in Y, the support functional of A is defined as $\sigma_A(y^*) = \sup_{y \in A} \{\langle y^*, y \rangle\}, y^* \in Y^*$.

Theorem 1　Let $S \subset X$ be a nonempty set and $F: S \to 2^Y$ be nearly E-subconvexlike on S. Then one and only one of the following statements is true: i) $\exists x \in S, F(x) \cap (-\mathrm{cor}E) \neq \varnothing$; ii) $\exists \mu \in K^+ \backslash \{0_{Y^*}\}$, $\langle \mu, y \rangle - \sigma_{-E}(\mu) \geqslant 0$, $\forall y \in F(S)$.

Proof　Assume that both i) and ii) hold, then there exists $x \in S$ such that $F(x) \cap -\mathrm{cor}E \neq \varnothing$. It follows from Lemma 3 that there exist $y \in F(x)$ and $e \in E$ such that $y + e \in -\mathrm{cor}K$. Hence from Lemma 2 there have $\langle \mu, y + e \rangle < 0$, i.e., $\langle \mu, y \rangle < \langle \mu, -e \rangle \leqslant \sup_{\tilde{e} \in -E} \langle \mu, \tilde{e} \rangle = \sigma_{-E}(\mu)$. Therefore, $\langle \mu, y \rangle - \sigma_{-E}(\mu) < 0$, which contradicts to ii).

If i) does not hold, then by Lemma 3, getting $(F(S) + E) \cap (-\mathrm{cor}K) = \varnothing$. Next, it first proves

$$\mathrm{cone}(F(S)+E)\cap(-\mathrm{cor}K)=\varnothing. \tag{3}$$

On the contrary, if $\mathrm{cone}(F(S)+E)\cap(-\mathrm{cor}K)\neq\varnothing$, then there exists $y\in Y\backslash\{0\}$ such that $y\in\mathrm{cone}(F(S)+E)$ and $y\in-\mathrm{cor}K$. Since $\mathrm{cone}(F(S)+E)$ and K are cones, then there exists $\lambda>0$ such that $\lambda y\in\mathrm{cone}(F(S)+E)$ and $\lambda y\in-\mathrm{cor}K$, which is a contradiction and hence (3) does hold.

Furthermore, the following will show

$$\mathrm{vcl}(\mathrm{cone}(F(S)+E))\cap(-\mathrm{cor}K)=\varnothing. \tag{4}$$

On the contrary, assuming $\mathrm{vcl}(\mathrm{cone}(F(S)+E))\cap(-\mathrm{cor}K)\neq\varnothing$, then there exists $y\in Y$ such that $y\in\mathrm{vcl}(\mathrm{cone}(F(S)+E))$ and $y\in-\mathrm{cor}K$. Since $y\in\mathrm{vcl}(\mathrm{cone}(F(S)+E))$, then from the definition of vector closure, it can obtain that there exists $h\in Y$, such that for any given $\varepsilon_1>0$, there exists $t_1\in(0,\varepsilon_1]$ satisfying $y+t_1h\in\mathrm{cone}(F(S)+E)$. Since $y\in-\mathrm{cor}K$ and $\mathrm{cor}K=\mathrm{cor}(\mathrm{cor}K)$, then there exists $\varepsilon_2>0$ such that $y+t_2h\in-\mathrm{cor}K$ for any $t_2\in[0,\varepsilon_2]$. Let $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$. Then there exists $t\in(0,\varepsilon]$ satisfying $y+th\in\mathrm{cone}(F(S)+E)$ and $y+th\in-\mathrm{cor}K$. This contradicts to $\mathrm{cone}(F(S)+E)\cap(-\mathrm{cor}K)=\varnothing$.

Moreover, from the fact that F is nearly E-subconvexlike on S, it follows that $\mathrm{vcl}(\mathrm{cone}(F(S)+E))$ is a convex set. Hence by Lemma 1, there exists $\mu\in Y^*\backslash\{0_{Y^*}\}$ such that

$$\langle\mu, y+e+\lambda k\rangle\geqslant 0,\ \forall y\in F(S),\ \forall e\in E,\ \forall k\in\mathrm{cor}K,\ \forall\lambda>0. \tag{5}$$

Let $\lambda\to+\infty$ in (5), then $\langle\mu,k\rangle\geqslant 0$, $\forall k\in\mathrm{cor}K$. Hence $\langle\mu,k\rangle\geqslant 0$, $\forall k\in K$ and so $\mu\in K^+\backslash\{0_{Y^*}\}$. Moreover, let $\lambda\to 0$ in (5), then $\langle\mu,y\rangle\geqslant\langle\mu,-e\rangle$, $\forall y\in F(S)$, $\forall e\in E$. Therefore, $\langle\mu,y\rangle\geqslant\sup\limits_{\tilde{e}\in-E}\langle\mu,e\tilde{e}\rangle=\sigma_{-E}(\mu)$, $\forall y\in F(S)$.

This implies ii) holds.

Remark 6 If Y is a real separated locally convex topological linear space, Y^* is the topological dual space of Y and $K\subset Y$ is a pointed closed convex cone with nonempty topological interior, then it is clear that above Theorem 1 coincides with Theorem 3.1[13].

3 Scalarization

In this section will establish scalarization theorem of weakly E-efficient solutions for vector optimization problems by using the alternative theorem with nearly E-subconvexlike set-valued map involving algebraic interior. Consider the following vector optimization problem:

$$(\mathrm{VP})\ \min_{x\in S}F(x),$$

where $S\subset X$, $S\neq\varnothing$ and $F:S\to 2^Y$ is a set-valued map with nonempty value.

Definition 3 A point $x_0\in S$ is called a weakly E-efficient solution of (VP) if there exists $y_0\in F(x_0)$ such that $(y_0-\mathrm{cor}E)\cap F(S)=\varnothing$. The point pair (x_0,y_0) is called a weakly E-efficient point of (VP).

Consider the following scalar optimization problem: $(\mathrm{VP})_\mu\ \min\limits_{x\in S}\langle\mu,F(x)\rangle$, $\mu\in Y^*\backslash\{0_{Y^*}\}$.

Definition 4[13] A point $x_0\in S$ is called an optimal solution of $(\mathrm{VP})_\mu$ with respect to E if there exists $y_0\in F(x_0)$ such that $\langle\mu,y-y_0\rangle\geqslant\sigma_{-E}(\mu)$, $\forall x\in S$, $\forall y\in F(x)$. The point pair (x_0,y_0) is called an optimal point of $(\mathrm{VP})_\mu$ with respect to E.

Theorem 2 Let $x_0\in S$, $y_0\in F(x_0)$ and $F-y_0$ be nearly E-subconvexlike on S. Then (x_0,y_0) is a weakly E-efficient point of (VP) if and only if there exists $\mu\in K^+\backslash\{0_{Y^*}\}$ such that (x_0,y_0) is an optimal point of $(\mathrm{VP})_\mu$ with respect to E.

Proof Assume that (x_0,y_0) is a weakly E-efficient point of (VP), then $(F(S)-y_0)\cap(-\mathrm{cor}E)=\varnothing$. Hence from Theorem 1, there exists $\mu\in K^+\backslash\{0_{Y^*}\}$ such that $\langle\mu,y-y_0\rangle-\sigma_{-E}(\mu)\geqslant 0$, $\forall y\in F(S)$. Therefore, $\langle\mu,y-y_0\rangle\geqslant\sigma_{-E}(\mu)$, $\forall x\in S$, $\forall y\in F(x)$.

Conversely, if (x_0,y_0) is not a weakly E-optimal point of (VP), then by making use of Lemma 3, there have $(y_0-E-\mathrm{cor}K)\cap F(S)\neq\varnothing$. Thus there exist $\hat{x}\in S$, $\hat{y}\in f(\hat{x})$ and $\hat{e}\in E$ such that $\hat{y}-y_0+\hat{e}\in-\mathrm{cor}K$. Since $\mu\in K^+\backslash\{0_{Y^*}\}$, then by Lemma 2, there have $\langle\mu,\hat{y}-y_0\rangle-\sigma_{-E}(\mu)\leqslant\langle\mu,\hat{y}-y_0+\hat{e}\rangle<0$, which contradicts to the fact that (x_0,y_0) is an optimal point of $(\mathrm{VP})_\mu$ with respect to E.

Remark 7 If Y is a real separated locally convex topological linear space, Y^* is the topological dual space of Y and $K \subset Y$ is a pointed closed convex cone with nonempty topological interior, then Theorem 2 coincides with Theorem 4.1 [13].

In the following, it present an example to illustrate Theorem 2.

Example 3　Let $X=\mathbf{R}, Y=\mathbf{R}^2$, $\mathscr{T}=\{A\times B\in\mathbf{R}^2 \mid A=(a,b), a\in\mathbf{R}, b\in\mathbf{R}, B=(-\infty,+\infty)\}$, $K=\mathbf{R}^2_+$, $E=K\setminus\{0\}$, $S=\mathbf{R}_+$ and $F(x)=\{(y_1,y_2)\in\mathbf{R}^2 \mid y_1=x, y_2\geqslant x\}$. Clearly,

$$K^+=\mathbf{R}^2_+, F(S)=\{(y_1,y_2)\mid y_1\geqslant 0, y_2\geqslant y_1\}.$$

It first verify that $(Y,\mathscr{T})$ is a real locally convex topological linear space. It is clear that $\mathscr{T}$ is a topology with convex neighborhood basis of zero $\mathscr{T}_0$, where

$$\mathscr{T}_0=\{A\times B\in\mathbf{R}^2 \mid A=(-a,a), a\in\mathbf{R}_{++}, B=(-\infty,+\infty)\}.$$

Moreover, $TV_1:(y_1,y_2)\mapsto y_1+y_2$ is a continuous map from $\mathbf{R}^2\times\mathbf{R}^2$ to $\mathbf{R}^2$. For any given neighborhood of zero U, there exists a convex neighborhood of zero $\overline{U}$ such that $2\overline{U}\subset U$. Therefore, for the neighborhood $(y_1,y_2)+\overline{U}\times\overline{U}$, there have $TV_1((y_1,y_2)+\overline{U}\times\overline{U})\subset y_1+y_2+2\overline{U}\subset y_1+y_2+U$, which implies that TV_1 is a continuous map. Furthermore, $TV_2:(\lambda,y)\mapsto\lambda y$ is a continuous map from $\mathbf{R}\times\mathbf{R}^2$ to $\mathbf{R}^2$. For any given neighborhood of zero U, there exists a convex neighborhood of zero $\overline{U}$ such that $2\overline{U}\subset U$. In addition, there exist $\delta>0$ and a neighborhood of zero V such that $\lambda V\subset\overline{U}$ and $\lambda'(y+V)\subset\overline{U}$ for all $\lambda'\in(-\delta,\delta)$. Therefore, for the neighborhood $(\lambda,y)+(-\delta,\delta)\times V$, there have

$$TV_2((\lambda,y)+(-\delta,\delta)\times V)\subset\lambda y+2\overline{U}\subset\lambda y+U,$$

which implies that TV_2 is a continuous map. Hence $(Y,\mathscr{T})$ is a real locally convex topological linear space. In this space, it can verify that $\mathrm{int}E=\mathrm{int}K=\varnothing$ and $\mathrm{cor}E=\mathrm{cor}K=\mathbf{R}^2_{++}$.

In the following, taking $x_0=0$, $y_0=(0,0)\in F(x_0)$ and $\mu=(1,0)\in K^+$. Obviously, $\mathrm{vcl}(\mathrm{cone}(F(S)-y_0+E))=\mathbf{R}^2_+$ is a convex set and hence $F-y_0$ is nearly E-subconvexlike on S. Moreover, $(y_0-\mathrm{cor}E)\cap F(S)=\varnothing$. This means that (x_0,y_0) is a weakly E-efficient point of (VP). Since $\sigma_{-E}(\mu)=\sup\limits_{\bar{e}\in -E}\langle\mu,\bar{e}\rangle=0$. Then $\langle\mu,y-y_0\rangle=y_1+0=x\geqslant 0=\sigma_{-E}(\mu)$, $\forall x\in S, y=(y_1,y_2)\in F(x)$.

Thus (x_0,y_0) is an optimal point of $(\mathrm{VP})_\mu$ with respect to E.

Remark 8　If $F-y_0$ is not nearly E-subconvexlike on S, then the conclusion is not necessarily true. The following example illustrates it.

Example 4　Let $X=Y=\mathbf{R}^2$, $F(x)=\{x\}$ and

$$K=\mathbf{R}^2_+, E=\mathbf{R}^2_+\setminus\{(y_1,y_2)\in\mathbf{R}^2 \mid 0\leqslant y_1<1, 0\leqslant y_2<1\},$$

$$S=\{(x_1,x_2)\in\mathbf{R}^2 \mid x_2=0, -3\leqslant x_1\leqslant 0\}\cup\{(x_1,x_2)\in\mathbf{R}^2 \mid x_1=0, -3\leqslant x_2\leqslant 0\}.$$

Clearly, $K^+=\mathbf{R}^2_+$, $F(S)=S$ and E is an improvement set with respect to K. Moreover, $\mathrm{cor}K=\mathbf{R}^2_{++}$ and $\mathrm{cor}E=\mathbf{R}^2_{++}\setminus\{(y_1,y_2)\in\mathbf{R}^2 \mid 0\leqslant y_1\leqslant 1, 0\leqslant y_2\leqslant 1\}$.

In the following, taking $x_0=(0,0)$, $y_0=(0,0)\in F(x_0)$. Obviously, $\mathrm{vcl}(\mathrm{cone}(F(S)-y_0+E))=\mathbf{R}^2\setminus(-\mathbf{R}^2_{++})$ is not a convex set and hence $F-y_0$ is not nearly E-subconvexlike on S. Since $(y_0-\mathrm{cor}E)\cap F(S)=\varnothing$, then (x_0,y_0) is a weakly E-efficient point of (VP). However, it can verify that for all $\mu=(\mu_1,\mu_2)\in K^+\setminus\{0\}$, the following implications hold:

i) If $\mu_1=\mu_2>0$, then there have $\sigma_{-E}(\mu)=-\mu_1$ and there eixsts $\bar{y}=(-2,0)\in F(S)$ such that

$$\langle\mu,\bar{y}-y_0\rangle=-2\mu_1<-\mu_1=\sigma_{-E}(\mu);$$

ii) If $\mu_1>\mu_2\geqslant 0$, then there have $\sigma_{-E}(\mu)=-\mu_2$ and there eixsts $\bar{y}=(-1,0)\in F(S)$ such that

$$\langle\mu,\bar{y}-y_0\rangle=-\mu_1<-\mu_2=\sigma_{-E}(\mu);$$

iii) $\mu_2>\mu_1\geqslant 0$, then there have $\sigma_{-E}(\mu)=-\mu_1$ and there eixsts $\bar{y}=(0,-1)\in F(S)$ such that

$$\langle\mu,\bar{y}-y_0\rangle=-\mu_2<-\mu_1=\sigma_{-E}(\mu).$$

Above all, (x_0,y_0) is not an optimal point of $(\mathrm{VP})_\mu$ with respect to E.

4 Lagrange multiplier theorem

In this section will establish Lagrange multiplier theorem of weakly E-efficient solutions for (VP).

Consider the case that $S=\{x\in D\mid G(x)\cap(-P)\neq\emptyset\}$, where $D\subset X$, $G:D\to 2^Z$ is a set-valued map with nonempty value and P is a positive cone with nonempty algebraic interior in Z, Let $L(Z,Y)$ be the set of all linear operators from Z to Y, A subset $L^+(Z,Y)$ of $L(Z,Y)$ is defined as

$$L^+=L^+(Z,Y)=\{T\in L(Z,Y)\mid T(P)\subset K\}.$$

Let $T\in L(Z,Y)$. Define $F+TG:S\to 2^Y$ by $(F+TG)(x)=F(x)+T(G(x))$. If there exists $\hat{x}\in S$ such that $G(\hat{x})\cap(-\mathrm{cor}P)\neq\emptyset$, then it says that (VP) satisfies the generalized Slater constraint qualification.

The Lagrangian function of (VP) $L:D\times L^+(Z,Y)\to Y$ is defined by

$$L(x,T):=F(x)+T(G(x)),(x,T)\in D\times L^+(Z,Y).$$

Theorem 3 Let $(F-y_0,G)$ be nearly $(E\times P)$-subconvexlike on D and (VP) satisfy the generalized Slater constraint qualification, If (x_0,y_0) is a weakly E-efficient point of (VP) and $0\in G(x_0)$, then there exists $T\in L^+$ such that (x_0,y_0) is a weakly E-optimal point of the following unconstrained vector optimization problem:

$$\text{(UVP)}\quad \min L(x,T),$$
$$\text{s. t. } (x,T)\in D\times L^+(Z,Y).$$

and $-T(G(x_0)\cap(-P))\subset(\mathrm{cor}K\cup\{0\})\backslash\mathrm{cor}E$.

Proof Since (x_0,y_0) is a weakly E-efficient point of (VP), then $x_0\in S$, $y_0\in F(x_0)$ and $(F(S)-y_0)\cap(-\mathrm{cor}E)=\emptyset$. Hence,

$$(F(D)-y_0,G(D))\cap(-\mathrm{cor}E,-P)=\emptyset. \tag{6}$$

From the nearly $(E\times P)$-subconvexlikeness on D of $(F-y_0,G)$ and by Theorem 1 and (6), there exists $(\mu,\varphi)\in K^+\times P^+\backslash\{(0_{Y^*},0_{Z^*})\}$ such that for any $x\in D, y\in F(x), z\in G(x), e\in E, z'\in P$.

$$\langle\mu,y-y_0\rangle+\langle\varphi,z\rangle\geqslant\sigma_{-E}(\mu)+\sigma_{-P}(\varphi)\geqslant\langle\mu,-e\rangle+\langle\varphi,-z'\rangle. \tag{7}$$

In particular, letting $z'=0$ in (7), it obtain that

$$\langle\mu,y-y_0+e\rangle+\langle\varphi,z\rangle\geqslant 0,\forall x\in D,\forall y\in F(x),\forall z\in G(x),\forall e\in E. \tag{8}$$

Since G satisfies the generalized Slater constraint qualification and by (8), there have $\mu\in K^+\backslash\{(0_{Y^*})\}$, Taking $k_0\in\mathrm{cor}K$ satisfying $\langle\mu,k_0\rangle=1$ and define $T:Z\to Y$ by

$$T(z)=\langle\varphi,z\rangle k_0,z\in Z. \tag{9}$$

Clearly, $T\in L^+(Z,Y)$, Taking $x=x_0$, $y=y_0$ and $z=z_0\in G(x_0)\cap(-P)$ in (8) and from $\varphi\in P^+$, it obtains that

$$-\langle\mu,e\rangle\leqslant\langle\varphi,z_0\rangle\leqslant 0. \tag{10}$$

By (10), there have $-T(z_0)=-\langle\varphi,z_0\rangle k_0\in\mathrm{cor}K\cup\{0\}$. By means of (10), there have $-T(z_0)\notin\mathrm{cor}E$. Otherwise, from Lemma 3, there exists $\bar{e}\in E$ such that $-T(z_0)-\bar{e}\in\mathrm{cor}K$. Consequently, $\langle\mu,T(z_0)+\bar{e}\rangle<0$, i.e., $\langle\varphi,z_0\rangle<-\langle\mu,\bar{e}\rangle$, which contradicts to (10). Noticing that z_0 is arbitrary in the set $G(x_0)\cap(-P)$, it obtains that $-T(G(x_0)\cap(-P))\subset(\mathrm{cor}K\cup\{0\})\backslash\mathrm{cor}E$. Furthermore, from $T\in L^+(Z,Y)$ and $0\in G(x_0)$, it follows that $0\in T(G(x_0))$. Thus, it gets that $y_0\in F(x_0)\subset F(x_0)+T(G(x_0))=L(x_0,T)$. Hence, from (8) and (9), it follows that for any $x\in S, y\in F(x), z\in G(x), e\in E$, $\langle\mu,y+T(z)\rangle=\langle\mu,y\rangle+\langle\varphi,z\rangle\langle\mu,k_0\rangle\geqslant\langle\mu,y_0-e\rangle$.

This implies that $\langle\mu,y+T(z)-y_0\rangle\geqslant\sigma_{-E}(\mu),\forall x\in S,\forall y\in F(x),\forall z\in G(x)$. Hence, (x_0,y_0) is an optimal point with respect to E for the problem $(\text{UVP})_\mu$ given by $(\text{UVP})_\mu \min_{(x,T)\in S\times L^+(Z,Y)}\langle\mu,L(x,T)\rangle$. It follows from Theorem 2 that (x_0,y_0) ia a weakly E-optimal point of (UVP).

Remark 9 If Y and Z are two real separated locally convex topological linear spaces, Y^* and Z^* are the topological dual space of Y and Z respectively, $K\subset Y$ and $P\subset Z$ are two pointed closed convex cones with nonempty topological interior, then above Theorem 3 coincides with Theorem 5.1 [13].

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运筹学与控制论

向量优化问题弱 E-有效解的代数性质

林 安[1]，朱 巧[2]，赵克全[2]
(1. 重庆师范大学涉外商贸学院 数学与计算机学院，重庆 401520；2. 重庆师范大学 数学科学学院，重庆 401331)

摘要：【目的】研究一类集值向量优化问题。【方法】利用代数内部这一概念，建立基于改进集而定义的集值映射邻近 E-次似凸性的择一性定理，进而应用该定理来研究集值向量优化问题。【结果】给出了基于代数内部和改进集而定义的弱 E-有效解的线性标量化结果和拉格朗日乘子定理，同时也给出了一些例子并对主要结果进行了解释。【结论】主要结果是对最近一些文献中相应结果的改进与推广。

关键词：集值向量优化问题；改进集；代数内部；邻近 E-次似凸性；弱 E-有效解；标量化

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