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运筹学与控制论

严格 α-预不变凸函数和半严格 α-预不变凸函数的梯度性质*

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摘要:【目的】讨论 α-预不变凸函数的一些梯度性质。【方法】在条件 C、条件 A 和适当的一些条件下更深入地研究了它的一些性质。【结果】这些性质包括严格 α-预不变凸函数的一个梯度性质和半严格 α-预不变凸函数的两个梯度性质。【结论】证明了函数是(严格的)半严格 α-预不变凸，当且仅当对于任意具有不同函数值的两点，它都满足严格不变凸性不等式。

关键词:严格 α-预不变凸函数;半严格 α-预不变凸函数;梯度
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1 预备知识

凸性和广义凸性在数理经济学、工程和优化理论中有着重要作用。因此，凸性和广义凸性是数学规划中一个重要的研究方向。近些年来，凸性概念得到了广泛地推广和使用。其中，凸函数的一个重要的推广是由 Hanson[1] 引入的不变凸函数。这个概念从优化的角度来看是非常有趣的，因为它提供了一个更广泛的思路去研究优化和数学规划问题。Ben-Israel 和 Mond[2] 引入了预不变凸函数；Weir 和 Mond[3]，Noor[4] 分别证明了预不变凸函数保留了凸函数的一些良好的性质。(半严格、严格)预不变凸函数也已经被 Yang[5-6] 研究过。此外 Noor[7] 引入并研究了另一类广义凸函数，称为 α-预不变凸函数，且已经证明了 α-预不变凸函数在广义凸规划和多目标优化中有着重要的应用。最近文献[8-9]引入了一类新的广义凸函数，这类函数与 α-预不变凸函数密切相关，被称为严格和半严格 α-预不变凸函数，并建立了这类函数的一些性质。

受文献[5-6]的研究工作以及广义 α-预不变凸函数概念的重要性的启发，本文进一步考虑严格和半严格 α-预不变凸函数，给出严格 α-预不变凸函数和半严格 α-预不变凸函数的一些梯度性质。

定义 1[7]　设 $\boldsymbol{y}\in S\subset\mathbf{R}^n$。称 S 在 $\boldsymbol{y}$ 点是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的 α-不变凸集，如果 $\forall\,\boldsymbol{x},\boldsymbol{y}\in S$，$\lambda\in[0,1]$，$\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})\in S$。

定义 2[7]　设 $S\subset\mathbf{R}^n$ 是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的 α-不变凸集，并设 $f:S\to\mathbf{R}$。称 f 是 α-预不变凸函数，如果 $f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))\leqslant\lambda f(\boldsymbol{y})+(1-\lambda)f(\boldsymbol{x})$，$\forall\,\boldsymbol{x},\boldsymbol{y}\in S,\lambda\in[0,1]$。

定义 3[8]　设 $S\subset\mathbf{R}^n$ 是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的 α-不变凸集，并设 $f:S\to\mathbf{R}$。称 f 是严格 α-预不变凸函数，如果 $\forall\,\boldsymbol{x},\boldsymbol{y}\in K,\boldsymbol{x}\neq\boldsymbol{y},f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))<\lambda f(\boldsymbol{y})+(1-\lambda)f(\boldsymbol{x})$，$\forall\,\lambda\in(0,1)$。

定义 4[9]　设 $S\subset\mathbf{R}^n$ 是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的 α-不变凸集，并设 $f:S\to\mathbf{R}$。称 f 是半严格 α-预不变凸函数，如果 $\forall\,\boldsymbol{x},\boldsymbol{y}\in K,f(\boldsymbol{x})\neq f(\boldsymbol{y}),f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))<\lambda f(\boldsymbol{y})+(1-\lambda)f(\boldsymbol{x})$，$\forall\,\lambda\in(0,1)$。

例 1　设 $S=[-1,6]\cup[-6,-2]$，$f(x)=\begin{cases}1,x=0\\0,x\neq0\end{cases}$，$\alpha(x,y)=1$ 及 $\eta(x,y)=\begin{cases}x-y,x\in[-1,6],y\in[-1,6]\\x-y,x\in[-6,-2],y\in[-6,-2]\\-4-y,x\in[-1,6],y\in[-6,-2]\\-y,x\in[-6,-2],y\in[-1,6],y\neq0\\\dfrac{1}{6}x,x\in[-6,-2],y=0\end{cases}$，则 f 是关于 α 和 η 的半严格 α-预不变凸函数。但是，若令 $x=-1,y=1$，

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$\lambda=\frac{1}{2}$,有 $f[y+\lambda\alpha(x,y)\eta(x,y)]=f(0)=1>0=\frac{1}{2}f(-1)+\frac{1}{2}f(1)=\frac{1}{2}f(x)+\frac{1}{2}f(y)$。即 f 不是关于 α 和 η 的 α-预不变凸函数。

条件 C[7] $\eta(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))=-\lambda\eta(\boldsymbol{x},\boldsymbol{y}),\eta(\boldsymbol{x},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))=(1-\lambda)\eta(\boldsymbol{x},\boldsymbol{y}),\forall\boldsymbol{x},\boldsymbol{y}\in S,\forall\lambda\in[0,1]$。

条件 D[7] $f(\boldsymbol{y}+\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\leqslant f(\boldsymbol{x}),\forall\boldsymbol{x},\boldsymbol{y}\in S$。

2 主要结果

本节将建立半严格 α-预不变凸函数和严格 α-预不变凸函数的梯度性质定理。下面先给出 α-预不变凸函数的一个结果。

定理 1 设 $S\subset\mathbf{R}^n$ 是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的非空 α-不变凸集,且 $f:S\to\mathbf{R}$ 是关于 α 和 η 的 α-预不变凸函数。假设 η 满足条件 C,α 满足条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$,对 $\forall\boldsymbol{x},\boldsymbol{y}\in S$,令 $g(\lambda)=f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x})),\forall\lambda\in[0,1]$。则 $\frac{g(\gamma)-g(0)}{\gamma}\leqslant\frac{g(\beta)-g(0)}{\beta},0<\gamma<\beta\leqslant1$。即:

$$\frac{f(\boldsymbol{y}+\gamma\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))-f(\boldsymbol{y})}{\gamma}\leqslant\frac{f(\boldsymbol{y}+\beta\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))-f(\boldsymbol{y})}{\beta},0<\gamma<\beta\leqslant1。$$

证明 对于 $\forall 0<\gamma<\beta\leqslant1$,令 $\boldsymbol{z}_\gamma=\boldsymbol{y}+\gamma\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{z}_\beta=\boldsymbol{y}+\beta\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}),\mu=1-\frac{\gamma}{\beta}$。

由条件 C 和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$,有:

$$\boldsymbol{z}_\beta+\mu\alpha(\boldsymbol{y},\boldsymbol{z}_\beta)\eta(\boldsymbol{y},\boldsymbol{z}_\beta)=\boldsymbol{y}+\beta\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})+\mu\alpha(\boldsymbol{y},\boldsymbol{y}+\beta\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\eta(\boldsymbol{y},\boldsymbol{y}+\beta\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))=$$
$$\boldsymbol{y}+\beta\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})-\mu\beta\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{y}+(\beta-\mu\beta)\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{y}+\gamma\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{z}_\gamma。$$

因而有 $g(\gamma)=f(\boldsymbol{z}_\gamma)=f(\boldsymbol{z}_\beta+\mu\alpha(\boldsymbol{y},\boldsymbol{z}_\beta)\eta(\boldsymbol{y},\boldsymbol{z}_\beta))\leqslant\mu f(\boldsymbol{y})+(1-\mu)f(\boldsymbol{z}_\beta)=\left(1-\frac{\gamma}{\beta}\right)g(0)+\frac{\gamma}{\beta}g(\beta)$,从而有 $\frac{g(\gamma)-g(0)}{\gamma}\leqslant\frac{g(\beta)-g(0)}{\beta}$。 证毕

定理 2 设 $S\subset\mathbf{R}^n$ 是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的非空 α-不变凸集,且 $f:S\to\mathbf{R}$ 可微。假设 η 满足条件 C,α 满足条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{x},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$。则 f 在 S 上是关于 α 和 η 的严格 α-预不变凸函数 $\Leftrightarrow f$ 关于相同的 α 和 η 是严格 α-不变凸函数,即:

$$\forall\boldsymbol{x},\boldsymbol{y}\in S,\boldsymbol{x}\neq\boldsymbol{y},f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})。$$

证明 假设 f 是严格 α-预不变凸函数。由定义 $\forall\boldsymbol{x},\boldsymbol{y}\in S,\boldsymbol{x}\neq\boldsymbol{y}$,有:

$$f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))<\lambda f(\boldsymbol{y})+(1-\lambda)f(\boldsymbol{x}),\forall\lambda\in(0,1)。$$

由此得 $\frac{f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))-f(\boldsymbol{x})}{\lambda}<f(\boldsymbol{y})-f(\boldsymbol{x}),\forall\lambda\in(0,1)$。

由定理 1,有 $\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})=\inf\limits_{\lambda\geqslant0}\frac{f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))-f(\boldsymbol{x})}{\lambda}<f(\boldsymbol{y})-f(\boldsymbol{x})$,即:

$$f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})。$$

反之,假设 $\forall\boldsymbol{x},\boldsymbol{y}\in S,\boldsymbol{x}\neq\boldsymbol{y},\lambda\in(0,1)$。由 f 的严格 α-不变凸性,有:

$$f(\boldsymbol{x})-f(\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))>$$
$$\alpha(\boldsymbol{x},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))^{\mathrm{T}}\nabla f(\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))。\quad(1)$$

类似地,将严格 α-不变凸性条件用到 $\boldsymbol{y},\boldsymbol{y}+\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})$,有:

$$f(\boldsymbol{y})-f(\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))>$$
$$\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))^{\mathrm{T}}\nabla f(\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))。\quad(2)$$

将(1)式乘以 λ,(2)式乘以 $(1-\lambda)$,再相加,可以得到:

$$\lambda f(\boldsymbol{x})+(1-\lambda)f(\boldsymbol{y})-f(\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))>(\lambda\alpha(\boldsymbol{x},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))^{\mathrm{T}}+$$
$$(1-\lambda)\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))^{\mathrm{T}})\nabla f(\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))。$$

然而,由条件 C 和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$ 得到:

$$\lambda\alpha(\boldsymbol{x},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\eta(\boldsymbol{x},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))+$$

$$(1-\lambda)\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\eta(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))=0。$$ 证毕

在给出半严格 α-预不变凸函数的梯度性质定理 3 和定理 4 之前，先给出下面的一个假设条件。

条件 A　设 $S\subset\mathbf{R}^n$ 是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的非空 α-不变凸集。称函数 f 满足条件 A，如果 $\forall\,\boldsymbol{x},\boldsymbol{y}\in S$，$f(\boldsymbol{x})<f(\boldsymbol{y})$，$f(\boldsymbol{y}+\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))<f(\boldsymbol{y})$。

例 2　设 $f(x)=-|x|$，$\forall\,x\in[-1,1]$，$\alpha(x,y)=1$，且 $\eta(x,y)=\begin{cases}x-y,x\geqslant 0,y\geqslant 0;x\leqslant 0,y\leqslant 0\\ y-x,x>0,y>0;x<0,y<0\end{cases}$，则 f 满足条件 A。

定理 3　设 $S\subset\mathbf{R}^n$ 是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的非空 α-不变凸集，且 η 满足条件 C，α 满足条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$。$f:S\to\mathbf{R}$ 可微且满足条件 A。则 f 在 S 上是关于 α 和 η 的半严格 α-预不变凸函数 $\Leftrightarrow\forall\,\boldsymbol{x},\boldsymbol{y}\in S$：$f(\boldsymbol{x})\neq f(\boldsymbol{y})$，$f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\,\nabla f(\boldsymbol{x})$。

证明　假设 f 是半严格 α-预不变凸函数。由定义，对 $\forall\,\boldsymbol{x},\boldsymbol{y}\in S$，$f(\boldsymbol{x})\neq f(\boldsymbol{y})$ 及 $f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))<\lambda f(\boldsymbol{y})+(1-\lambda)f(\boldsymbol{x})$，$\forall\,\lambda\in(0,1)$ 可得：$\dfrac{f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))-f(\boldsymbol{x})}{\lambda}<f(\boldsymbol{y})-f(\boldsymbol{x})$，$\forall\,\lambda\in(0,1)$。

根据文献[9]中的定理 1 和本文定理 1，有：

$$\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\,\nabla f(\boldsymbol{x})=\inf_{\lambda>0}\frac{f(\boldsymbol{x}+\lambda\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x})-f(\boldsymbol{x})}{\lambda}<f(\boldsymbol{y})-f(\boldsymbol{x}),$$

即 $f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\,\nabla f(\boldsymbol{x})$。

反之，假设 $\forall\,\boldsymbol{x},\boldsymbol{y}\in S$，$f(\boldsymbol{x})\neq f(\boldsymbol{y})$，$f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\,\nabla f(\boldsymbol{x})$。令 $\boldsymbol{z}_\gamma=\boldsymbol{y}+\gamma\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})$，$\forall\,\gamma\in(0,1)$。不失一般性，假设 $f(\boldsymbol{x})<f(\boldsymbol{y})$，需证 $f(\boldsymbol{z}_\gamma)\neq f(\boldsymbol{y})$，$\forall\,\gamma\in(0,1)$。

反证法。假设存在 $\gamma_0\in(0,1)$，有：

$$f(\boldsymbol{z}_{\gamma_0})=f(\boldsymbol{y})。\tag{3}$$

由(3)式，需证 $f(\boldsymbol{z}_{\gamma_0}+\lambda\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=f(\boldsymbol{y})$，$\forall\,\lambda\in(0,1)$。

假设存在 $\bar{\lambda}\in(0,1)$，有 $f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\neq f(\boldsymbol{y})$。

i) 若 $f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))>f(\boldsymbol{y})$，令 $g(\lambda)=f(\boldsymbol{z}_{\gamma_0}+\lambda\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))$，$\forall\,\lambda\in[0,1]$。因为 $g(0)=f(\boldsymbol{z}_{\gamma_0})=f(\boldsymbol{y})$，则由条件 C 和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$，有：

$$\begin{gathered}g(1)=f(\boldsymbol{z}_{\gamma_0}+\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=\\ f(\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})+\alpha(\boldsymbol{y},\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\eta(\boldsymbol{y},\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})))=\\ f(\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})-\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))=f(\boldsymbol{y})。\end{gathered}$$

从而，g 在(0,1)取得极大值。假设 g 在 $\lambda_0\in(0,1)$ 达到最大值，则：

$$0=g'(\lambda_0)=\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})^{\mathrm{T}}\,\nabla f(\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))。$$

由条件 C 和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$，有：

$$\alpha(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\,\nabla f(\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=0。$$

再次利用条件 C 和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$，可以得到：

$$\begin{gathered}\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=\\ \eta(\boldsymbol{y},\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\eta(\boldsymbol{y},\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})))=\\ \eta(\boldsymbol{y},\boldsymbol{y}+(\gamma_0-\lambda_0\gamma_0)\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))=-(\gamma_0-\lambda_0\gamma_0)\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}),\\ \alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=\\ \alpha(\boldsymbol{y},\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\eta(\boldsymbol{y},\boldsymbol{y}+\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})))=\\ \alpha(\boldsymbol{y},\boldsymbol{y}+(\gamma_0-\lambda_0\gamma_0)\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))=\alpha(\boldsymbol{x},\boldsymbol{y})。\end{gathered}$$

从而有：

$$\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))^{\mathrm{T}}\,\nabla f(\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=0。\tag{4}$$

根据 $f(\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\geqslant f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\geqslant f(\boldsymbol{y})$、(4)式和定理的假设条件，有：

$$\begin{gathered}f(\boldsymbol{y})>f(\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))+\\ \alpha^{\mathrm{T}}(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta^{\mathrm{T}}(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\nabla f(\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=\\ f(\boldsymbol{z}_{\gamma_0}+\lambda_0\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=g(\lambda_0)。\end{gathered}$$

这与 g 在 λ_0 点取得最大值假设矛盾。

ii) 若 $f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))<f(\boldsymbol{y})$。由 $f(\boldsymbol{z}_{\gamma_0})=f(\boldsymbol{y})$，$f(\boldsymbol{x})<f(\boldsymbol{y})$ 和条件A，有：

$$f(\boldsymbol{y}+\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))<f(\boldsymbol{y})=f(\boldsymbol{z}_{\gamma_0})。$$

令 $g(\lambda)=f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\lambda\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))$。因为 $g(0)=f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))<f(\boldsymbol{y})=f(\boldsymbol{z}_{\gamma_0})$，根据条件C、条件A和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\cdot\eta(\boldsymbol{x},\boldsymbol{y}))$，可以得到：

$$g(1)=f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))=$$
$$f(\boldsymbol{y}+\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))<f(\boldsymbol{y})=f(\boldsymbol{z}_{\gamma_0})。$$

因为 $0<\dfrac{\gamma_0\bar{\lambda}}{1-\gamma_0(1-\bar{\lambda})}<1$，由条件C和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$，有 $g\left(\dfrac{\gamma_0\bar{\lambda}}{1-\gamma_0(1-\bar{\lambda})}\right)=f(\boldsymbol{z}_{\gamma_0})$。从而，$g(\lambda)=f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\lambda\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))$ 在 $(0,1)$ 取得最大值。假设在 $\lambda_0\in(0,1)$ 达到最大值，则：

$$0=g'(\lambda_0)=\alpha^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\cdot$$
$$\nabla f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))。$$

因为：

$$f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))>$$
$$f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))，$$

所以有：

$$f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))>$$
$$f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))+$$
$$\alpha^{\mathrm{T}}(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}),\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+$$
$$\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))\cdot$$
$$\eta^{\mathrm{T}}(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}),\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\cdot$$
$$\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))\nabla f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+$$
$$\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))=$$
$$f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\lambda_0\alpha^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))-$$
$$\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))^{\mathrm{T}}+$$
$$\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))\nabla f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+$$
$$\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))=$$
$$f(\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})+\lambda_0\alpha(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))\eta(\boldsymbol{x},\boldsymbol{z}_{\gamma_0}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})))。$$

这与 $g(\lambda)$ 在 $\lambda_0\in(0,1)$ 达到最大值的假设矛盾。

结合 i)和 ii)，有：

$$f(\boldsymbol{z}_{\gamma_0}+\lambda\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))=f(\boldsymbol{y}),\ \forall\lambda\in[0,1]。\tag{5}$$

令 $h(\lambda)=f(\boldsymbol{z}_{\gamma_0}+\lambda\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0}))$。则由(5)式得到：

$$0=h'(1)=\alpha(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{z}_{\gamma_0})^{\mathrm{T}}\nabla f(\boldsymbol{y})=-\gamma_0\alpha(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\nabla f(\boldsymbol{y}),$$

即：

$$\alpha(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\nabla f(\boldsymbol{y})=0。\tag{6}$$

根据定理的假设条件和(4)式，有 $f(\boldsymbol{x})>f(\boldsymbol{y})+\alpha(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\nabla f(\boldsymbol{y})=f(\boldsymbol{y})$，这与 $f(\boldsymbol{x})<f(\boldsymbol{y})$ 矛盾。所以 $f(\boldsymbol{z}_\gamma)\neq f(\boldsymbol{y}),\ \forall\gamma\in(0,1)$。

若 $\exists\gamma\in(0,1)$，$f(\boldsymbol{z}_\gamma)=f(\boldsymbol{x})$，则由 $f(\boldsymbol{x})<f(\boldsymbol{y})$，有 $f(\boldsymbol{z}_\gamma)<\gamma f(\boldsymbol{x})+(1-\gamma)f(\boldsymbol{y})$。

若 $\exists\gamma\in(0,1)$，$f(\boldsymbol{z}_\gamma)\neq f(\boldsymbol{x})$，根据定理的假设条件，以及条C和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$，可以得到：

$$f(\boldsymbol{x})>f(\boldsymbol{z}_\gamma)+\alpha^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{z}_\gamma)\eta^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{z}_\gamma)\nabla f(\boldsymbol{z}_\gamma)=f(\boldsymbol{z}_\gamma)+(1-\gamma)\alpha^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{y})\eta^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{y})\nabla f(\boldsymbol{z}_\gamma), \tag{7}$$

$$f(\boldsymbol{y})>f(\boldsymbol{z}_\gamma)+\alpha^{\mathrm{T}}(\boldsymbol{y},\boldsymbol{z}_\gamma)\eta^{\mathrm{T}}(\boldsymbol{y},\boldsymbol{z}_\gamma)\nabla f(\boldsymbol{z}_\gamma)=f(\boldsymbol{z}_\gamma)-\gamma\alpha^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{y})\eta^{\mathrm{T}}(\boldsymbol{x},\boldsymbol{y})\nabla f(\boldsymbol{z}_\gamma)。 \tag{8}$$

用 γ 乘以(7)式,$(1-\gamma)$乘以(8)式,再将它们相加,有 $f(\boldsymbol{z}_\gamma)<\gamma f(\boldsymbol{x})+(1-\gamma)f(\boldsymbol{y})$。 证毕

定理 4 设 $S\subset\mathbf{R}^n$ 是关于 $\eta:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ 和 $\alpha:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ 的非空 α-不变凸集,且 η 满足条件 C,α 满足条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$。$f:S\to\mathbf{R}$ 可微且满足条件 A 和条件 $f(\boldsymbol{x}+\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))=f(\boldsymbol{y})$。则 f 在 S 上是关于 α 和 η 的半严格 α-预不变凸函数 $\Leftrightarrow\forall\boldsymbol{x},\boldsymbol{y}\in S$, $f(\boldsymbol{x})\neq f(\boldsymbol{y})$, $\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})+\alpha(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\nabla f(\boldsymbol{y})<0$。

证明 假设 f 是半严格α-预不变凸函数。令 $\boldsymbol{x},\boldsymbol{y}\in S$, $f(\boldsymbol{x})\neq f(\boldsymbol{y})$,根据定理 3,可得:

$$f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}), \tag{9}$$

$$f(\boldsymbol{x})>f(\boldsymbol{y})+\alpha(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\nabla f(\boldsymbol{y})。 \tag{10}$$

由(9),(10)式有 $\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})+\alpha(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\nabla f(\boldsymbol{y})<0$。

反之,假设 $\forall\boldsymbol{x},\boldsymbol{y}\in S$, $f(\boldsymbol{x})\neq f(\boldsymbol{y})$,有:

$$\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})+\alpha(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{y})^{\mathrm{T}}\nabla f(\boldsymbol{y})<0。 \tag{11}$$

根据定理 3,需证 $f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})$。

反证法。假设 $\exists\boldsymbol{x},\boldsymbol{y}\in S$, $f(\boldsymbol{x})\neq f(\boldsymbol{y})$,使得:

$$f(\boldsymbol{y})\leqslant f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}), \tag{12}$$

则由假设条件 $f(\boldsymbol{x}+\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))=f(\boldsymbol{y})$,得到:

$$f(\boldsymbol{x}+\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))=f(\boldsymbol{y})\leqslant f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}), \tag{13}$$

根据中值定理,有:

$$f(\boldsymbol{x}+\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))-f(\boldsymbol{x})=\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))。 \tag{14}$$

从而,由(13),(14)式得:

$$\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))\leqslant\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})。 \tag{15}$$

i) 若 $f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))\neq f(\boldsymbol{x})$,则由(11)式有:

$$\alpha(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})+$$
$$\alpha(\boldsymbol{x},\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))^{\mathrm{T}}\eta(\boldsymbol{x},\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))^{\mathrm{T}}\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))<0。$$

根据条件 C 和条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$,得到:

$$\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})<\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x})),$$

这与(15)式矛盾。

ii) 若 $f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))=f(\boldsymbol{x})$,需证 $\exists\alpha\in(0,\bar{\lambda})$, $f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))=f(\boldsymbol{x})\neq f(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\cdot\eta(\boldsymbol{y},\boldsymbol{x}))$。若其不然,即:

$$f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))=f(\boldsymbol{x})=f(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x})),\ \forall\alpha\in(0,\bar{\lambda})。 \tag{16}$$

令 $\varphi(\gamma)=f(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))$, $\forall\alpha\in[0,\bar{\lambda}]$,则由(16)式得 $\varphi(\gamma)=\mathrm{const}=f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))$。于是 $0=\varphi'(\gamma)=\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))$。从而 $\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))=0$。此与(14)式和 $f(\boldsymbol{x})\neq f(\boldsymbol{y})=f(\boldsymbol{x}+\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))$矛盾。所以有:

$$\exists\alpha\in(0,\bar{\lambda}),\ f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))=f(\boldsymbol{x})\neq f(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))。 \tag{17}$$

由(11),(17)式,可得:

$$\alpha(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}})\eta(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}})\cdot$$
$$\nabla f(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))+\alpha(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))^{\mathrm{T}}\cdot$$
$$\eta(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))^{\mathrm{T}}\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))<0,$$

和

$$\alpha(\boldsymbol{x},\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}})\eta(\boldsymbol{x},\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}})\nabla f(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))+$$
$$\alpha(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))^{\mathrm{T}}\nabla f(\boldsymbol{x})<0。$$

据条件 C,条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$和上面两个等式,可以得到:

$$\alpha(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))+$$
$$\alpha(\boldsymbol{x},\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}})\eta(\boldsymbol{x},\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}})\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))<0, \tag{18}$$

和

$$\alpha(\boldsymbol{x},\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}})\eta(\boldsymbol{x},\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}})\nabla f(\boldsymbol{x}+\gamma\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))+ \alpha(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}),\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})<0。\tag{19}$$

再次利用条件 C、条件 $\alpha(\boldsymbol{x},\boldsymbol{y})=\alpha(\boldsymbol{y},\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))$，并将(18)，(19)式相加，可以得到：

$$\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x}+\bar{\lambda}\alpha(\boldsymbol{y},\boldsymbol{x})\eta(\boldsymbol{y},\boldsymbol{x}))>\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})。\tag{20}$$

结合(14)，(20)式，得到 $f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})$，这与(12)式矛盾。从而，$\forall \boldsymbol{x},\boldsymbol{y}\in S$，$f(\boldsymbol{x})\neq f(\boldsymbol{y})$，$f(\boldsymbol{y})>f(\boldsymbol{x})+\alpha(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}\nabla f(\boldsymbol{x})$。

根据定理 3，f 是半严格 α-预不变凸函数。 证毕

例 3 令 $f(x)=\begin{cases}-1,x\neq 0\\ 0,x=0\end{cases}$，$\alpha(x,y)=1$，$\eta(x,y)=\begin{cases}x-y,x\geqslant 0,y\geqslant 0\\ x-y,x\leqslant 0,y\leqslant 0\\ 1-y,x<0,y\geqslant 0\\ -1-y,x>0,y\geqslant 0\end{cases}$，易知 f，η 和 α 分别满足条件 A、条件 C 及条件 $\alpha(x,y)=\alpha(y,y+\lambda\alpha(x,y)\eta(x,y))$ 和条件 $f(x+\alpha(y,x)\eta(y,x))=f(y)$。

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Operations Research and Cybernetics

Gradient Properties of Strictly α-Inexpressive and Semi Strictly α-Inexpressive Functions

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Abstract: [**Purposes**] α-inexpressive is a new kind of generalized convexity introduced by scholars. So the research focused on is the classes of α-preinvex function. [**Methods**] Under the Condition C, Condition A and some suitable conditions, some of its properties is studied more deeply. [**Findings**] These properties include a gradient criterion of strictly α-preinvex functions and two gradient properties of semistrictly α-preinvex functions. [**Conclusions**] It is shown that a function is (strictly) semistrictly α-preinvex if and only if it satisfies a strict invexity inequality for any two points with distinct function values.

Keywords: strictly α-preinvex functions; semistrictly α-preinvex functions; gradient

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