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Cahn-Hilliard-Navier-Stokes 方程的长时间行为*

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摘要:【目的】Cahn-Hilliard-Navier-Stokes 方程是由不可压缩的 Navier-Stokes 方程和高阶各项异性 Cahn-Hilliard 方程耦合而成,具有广泛应用和重要作用。【方法】通过能量估计得到方程的解半群存在有界吸收集和一致紧性。【结果】利用吸引子存在性定理验证了整体吸引子的存在性。【结论】研究了该方程在相对浓度满足 Neumann 边界条件下的长时间行为。

关键词: Cahn-Hilliard-Navier-Stokes 方程;Neumann 边界条件;吸收集;整体吸引子

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本文主要考虑如下 Cahn-Hilliard-Navier-Stokes 方程组的初边值问题:

$$\partial_t \boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\rho\Delta\boldsymbol{u}+\nabla p=\kappa\mu\nabla\varphi+\boldsymbol{g},\boldsymbol{x}\in\Omega,t>0,\tag{1}$$

$$\mathrm{div}\,\boldsymbol{u}=0,\boldsymbol{x}\in\Omega,t>0,\tag{2}$$

$$\partial_t\varphi+\boldsymbol{u}\cdot\nabla\varphi-\Delta\mu=0,\boldsymbol{x}\in\Omega,t>0,\tag{3}$$

$$\mu=\sum_{i=1}^{k}(-1)^i\sum_{|\alpha|=i}a_\alpha D^{2\alpha}\varphi+f(\varphi),\boldsymbol{x}\in\Omega,t>0,\tag{4}$$

$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0,\varphi(\boldsymbol{x},0)=\varphi_0,\boldsymbol{x}\in\Omega,\tag{5}$$

$$\boldsymbol{u}=0,\frac{\partial D^{2\alpha}\varphi}{\partial\boldsymbol{n}}=0,|\alpha|\leqslant k-1,\boldsymbol{x}\in\partial\Omega,t>0。\tag{6}$$

其中:$\Omega\subset\mathbf{R}^2$ 为有界区域,未知函数 $\boldsymbol{u}$、p 和 φ 分别表示流体速度、压强和相对浓度,已知函数 $\boldsymbol{g}$ 表示外力,正常数 ρ 和 κ 分别表示粘度系数和应力系数,$\boldsymbol{n}$ 是边界的单位外法向量,k 是不小于 2 的整数,∇,Δ 分别表示梯度算子和 Laplace 算子,$D^\alpha=\dfrac{\partial^\alpha}{\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}}$,$\alpha=(k_1,k_2)\in(\mathbf{N}\cup\{0\})^2$,$|\alpha|=k_1+k_2$,规定 $D^{(0,0)}\varphi=\varphi$,a_α 是给定常数,当 $|\alpha|=k$ 时,$a_\alpha\geqslant 0$。

Cahn-Hilliard-Navier-Stokes 方程由 Cahn-Hilliard 方程和不可压缩的 Navier-Stokes 方程耦合而成,是一种重要的物理模型,应用广泛。其中 Cahn-Hilliad 方程是基于 Ginzburg-Landau 自由能建立的,它的自由能为:$\Psi_{\mathrm{HOGL}}(\varphi)=\int_\Omega \frac{1}{2}\sum_{i=1}^{k}\sum_{|\alpha|=i}a_\alpha|D^\alpha\varphi|^2+F(\varphi)\mathrm{d}\boldsymbol{x}$,$k\in\mathbf{N},k\geqslant 2$。式(4)中 μ 是该自由能的变分导数,f 是 F 的导数,即:$f(s)=F'(s)$。

对于各向同性 Cahn-Hilliard-Navier-Stokes 方程的研究,在理论分析和数值计算方面都取得很大进展[1-11]。近些年来人们对高阶 Cahn-Hilliard-Navier-Stokes 方程也有一定的研究[12-16],如 Cherfils 等人[12]证明了高阶 Cahn-Hilliard-Navier-Stokes 方程弱解的存在唯一性,全局吸引子与指数吸引子的存在性,但是对高阶各向异性 Cahn-Hilliard-Navier-Stokes 方程则只有少量结果。罗娇等人[17-18]研究了高阶各向异性 Cahn-Hilliard-Navier-Stokes 方程在流体速度和相对浓度均满足 Dirichlet 边界条件下弱解的存在性和吸引子的存在性。基于此,本文主要研究各向异性 Cahn-Hilliard-Navier-Stokes 方程在流体速度满足 Dirichlet 边界而相对浓度满足 Neumann 边界条件下的长时间行为。文献[17-18]考虑相对浓度满足 Dirichlet 边界条件时,由 Poincaré 不等式可以得到 Sobolev 空间的范数和高阶导数在 L^2 空间的范数具有等价性,所以在空间和范数的引入相对自然,而在相对浓度满足 Neumann 边界条件下,Poincaré 不等式已经失效,主要困难在于空间和范数的处理,所以本文结果拓展

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了文献[17-18]的结论，丰富了 Cahn-Hilliard-Navier-Stokes 方程的研究结果。文中出现的 c、c_i、C 和 C_i 均表示可变的常数。

1　预备知识

假设 1　假设 $f\in C^2(\mathbf{R})$，且满足 $\liminf\limits_{|s|\to+\infty} f'(s)>0$，$|f''(s)|\leqslant c_f(1+|s|^{m-1})$，$\forall s\in\mathbf{R}$，其中 $c_f>0$，m 是 $[1,+\infty)$ 中的常数。

设 X 为实希尔伯特空间，X^* 表示 X 的对偶，$(\cdot,\cdot)_X$ 和 $|\cdot|_X$ 分别表示 X 的内积和范数，$\langle\cdot,\cdot\rangle$ 代表 X^* 与 X 的对偶积。设 $\Xi=\{\boldsymbol{u}\in C_c^\infty(\Omega,\mathbf{R}^2):\operatorname{div}\boldsymbol{u}=0,\ \boldsymbol{x}\in\Omega\}$，$H$ 为 Ξ 在 $L^2(\Omega,\mathbf{R}^2)$ 中的闭包，V 为 Ξ 在 $\mathrm{H}_0^1(\Omega,\mathbf{R}^2)$ 中的闭包，$(\cdot,\cdot)$ 和 $|\cdot|$ 分别表示 H 的内积和范数，空间 V 的内积和范数分别为 $((\boldsymbol{u},\boldsymbol{v}))=\sum\limits_{i=1}^{2}(\partial_{x_i}\boldsymbol{u},\partial_{x_i}\boldsymbol{v})$，$\|\boldsymbol{u}\|=((\boldsymbol{u},\boldsymbol{u}))^{\frac{1}{2}}$。记 $\langle\varphi\rangle=\dfrac{1}{|\Omega|}\int_\Omega\varphi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}$，由式(3)知对 $\forall t>0$ 有 $\langle\varphi(\boldsymbol{x},t)\rangle=\langle\varphi_0(\boldsymbol{x})\rangle=M_0$。记希尔伯特空间 $Y_M=H\times\{\varphi\in H^k(\Omega):|\langle\varphi\rangle|\leqslant M\}$，且赋予范数 $\|(\boldsymbol{u},\varphi)\|_{Y_M}^2=\dfrac{1}{\boldsymbol{\kappa}}|\boldsymbol{u}|^2+\sum\limits_{|\alpha|=k}a_\alpha|D^\alpha\varphi|_{L^2}^2+\langle\varphi\rangle^2$。

首先给出方程(1)～(6)弱解的定义。

定义 1　若 $k\in\mathbf{N}$，$k\geqslant 2$，初值 $(\boldsymbol{u}_0,\varphi_0)\in Y_M$，则存在 $(\boldsymbol{u},\varphi)\in L^\infty([0,+\infty);Y_0)\cap L^2([t,t+1];V\times H^{2k})$，$\partial_t\boldsymbol{u}\in L^2([t,t+1];V^*)$，$\partial_t\varphi\in L^2([t,t+1];L^2(\Omega))$，$\mu\in L^2([t,t+1];L^2(\Omega))$ 且满足：

$$(\partial_t\boldsymbol{u},\boldsymbol{v})+\rho(\nabla\boldsymbol{u},\nabla\boldsymbol{v})+((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v})=\boldsymbol{\kappa}(\mu\nabla\varphi,\boldsymbol{v})+(\boldsymbol{g},\boldsymbol{v}),\boldsymbol{v}\in V,\tag{7}$$

$$(\partial_t\varphi,\psi)+((\boldsymbol{u}\cdot\nabla)\varphi,\psi)+(\nabla\mu,\nabla\psi)=0,\psi\in L^2(\Omega),\tag{8}$$

$$(\mu,\xi)=\sum_{i=1}^{k}\sum_{|\alpha|=i}a_\alpha(D^\alpha\varphi,D^\alpha\xi)+(f(\varphi),\xi),\xi\in H_0^k(\Omega),\tag{9}$$

$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0,\varphi(\boldsymbol{x},0)=\varphi_0,\boldsymbol{x}\in\Omega。$$

则称 $(\boldsymbol{u},\varphi)$ 为方程(1)～(6)的弱解。

2　弱解的存在唯一性

在证明该方程的吸引子存在性之前，先给出弱解的存在唯一性。

定理 1　若 $k\in\mathbf{N}$，$k\geqslant 2$，$\boldsymbol{g}\in V^*$，$(\boldsymbol{u}_0,\varphi_0)\in Y_M$，$f$ 满足假设 1，则方程(1)～(6)存在唯一弱解 $(\boldsymbol{u},\varphi)$，对 $\forall T>0$，$(\boldsymbol{u},\varphi)\in L^\infty(\mathbf{R}^+,Y_M)\cap L^2(0,T;H_0^1(\Omega)\times(H_0^k(\Omega)\cap H^{2k}(\Omega)))$。

利用 Galerkin 方法可直接得到该定理，证明详细过程参见文献[17]。

由定理 1 显然可以得到如下定理。

定理 2　若 $k\in\mathbf{N}$，$k\geqslant 2$，$\boldsymbol{g}\in V^*$，$(\boldsymbol{u}_0,\varphi_0)\in Y_M$，$f$ 满足假设 1，则方程(1)～(6)定义了一个强连续半群 $S(t):Y_M\to Y_M$，对 $\forall t>0$，$S(t)(\boldsymbol{u}_0,\varphi_0)=(\boldsymbol{u},\varphi)$，其中 $(\boldsymbol{u},\varphi)$ 是方程(1)～(6)的弱解。

3　主要结论

定理 3　若 $k\in\mathbf{N}$，$k\geqslant 2$，$\boldsymbol{g}\in V^*$，f 满足假设 1，则 $S(t)$ 在 Y_M 存在有界吸收集，即对 $\forall\Sigma\subset Y_M$，存在 $t_0=t_0(\Sigma)$，使得 $\|S(t)(\boldsymbol{u}_0,\varphi_0)\|_{Y_M}\leqslant C$，$\forall t\geqslant t_0$。

证明　令式(7)中 $\boldsymbol{v}=\boldsymbol{\kappa}^{-1}\boldsymbol{u}$，式(8)中的 $\psi=\mu$ 和式(9)中的 $\xi=\partial_t\varphi$ 有：

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|(\boldsymbol{u}(t),\varphi(t))\|_{Y_M}^2+\sum_{i=1}^{k-1}\sum_{i=|\alpha|}|D^\alpha\varphi|_{L^2}^2+2(F(\varphi(t)),1)\Big]+\frac{2\rho}{\boldsymbol{\kappa}}\|\boldsymbol{u}(t)\|^2+2|\nabla\mu(t)|_{L^2}^2=\frac{2}{\boldsymbol{\kappa}}\langle\boldsymbol{g},\boldsymbol{u}(t)\rangle。\tag{10}$$

令 $\overline{\varphi}(t)=\varphi(t)-M_0$，$\overline{\mu}(t)=\mu(t)-\langle\mu(t)\rangle$。在式(4)两边乘以 $2\eta\overline{\varphi}$ 在 Ω 上积分得：

$$2\eta(\overline{\mu}(t),\overline{\varphi}(t))_{L^2}=2\eta\sum_{i=1}^{k}\sum_{|\alpha|=i}a_\alpha|D^\alpha\overline{\varphi}|^2+2\eta(f(\varphi(t)),\overline{\varphi}(t))_{L^2}。\tag{11}$$

由式(10)、(11)可得：

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|(\boldsymbol{u}(t),\varphi(t))\|_{Y_M}^2+\sum_{i=1}^{k-1}\sum_{i=|\alpha|}a_\alpha|D^\alpha\varphi|^2+2(F(\varphi(t)),1)\Big]+\frac{2\rho}{\boldsymbol{\kappa}}\|\boldsymbol{u}(t)\|^2+2|\nabla\mu(t)|_{L^2}^2=$$

$$\frac{2}{\kappa}\langle \boldsymbol{g},\boldsymbol{u}(t)\rangle+2\eta(\overline{\mu}(t),\overline{\varphi}(t))_{L^2}-2\eta\sum_{i=1}^{k}\sum_{|\alpha|=i}a_\alpha|D^\alpha\overline{\varphi}|^2-2\eta(f(\varphi(t)),\overline{\varphi}(t))_{L^2}。\tag{12}$$

令 $\Psi(t)=\|(\boldsymbol{u}(t),\varphi(t))\|_{Y_M}^2+\sum_{i=1}^{k-1}\sum_{i=|\alpha|}a_\alpha|D^\alpha\varphi|^2+2(F(\varphi(t)),1)+C_\Psi$，其中 C_Ψ 是足够大的常数，使得 $\Psi(t)$ 非负。于是，可以将式(12)写成：

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)+\varepsilon\Psi(t)=\Pi(t),\tag{13}$$

式中：$0<\varepsilon<\eta$，且

$$\Pi(t)=-\frac{2\rho}{\kappa}\|\boldsymbol{u}(t)\|^2-2|\nabla\mu(t)|_{L^2}^2+\frac{2}{\kappa}\langle \boldsymbol{g},\boldsymbol{u}(t)\rangle+2\eta,(\overline{\mu}(t),\overline{\varphi}(t))_{L^2}-2\eta\sum_{i=1}^{k}\sum_{|\alpha|=i}a_\alpha|D^\alpha\overline{\varphi}|_{L^2}{}^2-$$
$$2\eta(f(\varphi(t)),\overline{\varphi}(t))_{L^2}+\varepsilon\left[\|(\boldsymbol{u}(t),\varphi(t))\|_{Y_0}^2+\sum_{i=1}^{k-1}\sum_{i=|\alpha|}a_\alpha|D^\alpha\varphi|_{L^2}^2+2(F(\varphi(t)),1)+C_\Psi\right]。$$

即：

$$\Pi(t)=-\frac{2\rho}{\kappa}\|\boldsymbol{u}(t)\|^2-2|\nabla\mu(t)|_{L^2}^2+\frac{\varepsilon\rho}{\kappa}|\boldsymbol{u}(t)|^2+2\eta(\overline{\mu}(t),\overline{\varphi}(t))_{L^2}-(2\eta-\varepsilon)\sum_{i=1}^{k}\sum_{|\alpha|=i}a_\alpha|D^\alpha\overline{\varphi}|^2-$$
$$2(\eta-\varepsilon)(f(\varphi(t)),\overline{\varphi}(t))_{L^2}+2\varepsilon(F(\varphi(t))-f(\varphi(t))\overline{\varphi}(t),1)_{L^2}+\frac{2}{\kappa}\langle \boldsymbol{g},\boldsymbol{u}(t)\rangle+\varepsilon C_\Psi。$$

下面给出 $\Pi(t)$ 的估计。由 Hölder 不等式、Friedrich 不等式和 Young 不等式可得：

$$2\eta|(\overline{\mu}(t),\overline{\varphi}(t))_{L^2}|\leqslant 2\eta|\overline{\mu}(t)|_{L^2}|\overline{\varphi}(t)|_{L^2}\leqslant 2\eta C_\Omega|\nabla\overline{\mu}(t)|_{L^2}|\overline{\varphi}(t)|_{L^2}\leqslant|\nabla\overline{\mu}(t)|_{L^2}^2+\eta^2C_\Omega^2|\overline{\varphi}(t)|_{L^2}^2,\tag{14}$$

又因为 f 满足假设 1，所以对 $\forall s\in\mathbf{R}$ 有：

$$c_*|f(s)|(1+|s|)\leqslant 2f(s)(s-M_0)+c_{f,M_0},F(s)-f(s)(s-M_0)\leqslant c_f'|s-M_0|^2+c_{f,M_0}'',\tag{15}$$

式中：c_*、c_{f,M_0}、c_f'、c_{f,M_0}'' 均为非负常数。

由 Nirenberg-Gagliardo 不等式[19]可得：

$$\left|\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha|D^\alpha\overline{\varphi}|_{L^2}^2\right|\leqslant\varepsilon_1|\overline{\varphi}(t)|_{H^k}^2+c(\varepsilon_1)|\overline{\varphi}(t)|_{L^2}^2+\cdots+\varepsilon_{k-1}|\overline{\varphi}(t)|_{H^k}^2+c(\varepsilon_{k-1})|\overline{\varphi}(t)|_{L^2}^2,$$

当 $\varepsilon_i(i=1,2,\cdots,k-1)$ 取合适的值时有：

$$\left|\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha|D^\alpha\overline{\varphi}|_{L^2}^2\right|\leqslant\frac{1}{2}\sum_{|\alpha|=k}a_\alpha|D^\alpha\overline{\varphi}|_{L^2}^2+c|\overline{\varphi}(t)|_{L^2}^2。\tag{16}$$

根据 Hölder 不等式和 Young 不等式有：

$$|\langle\boldsymbol{u}(t),\boldsymbol{g}\rangle|\leqslant\|\boldsymbol{u}(t)\|\|\boldsymbol{g}\|_{V^*}\leqslant\frac{\rho}{2}\|\boldsymbol{u}(t)\|^2+\frac{1}{2\rho}\|\boldsymbol{g}\|_{V^*}^2。\tag{17}$$

结合式(13)～(17)和 Poincaré 不等式：可以得到：

$$\Pi(t)\leqslant-\frac{\rho}{\kappa}\|\boldsymbol{u}(t)\|^2+\frac{\varepsilon\rho}{\kappa}|\boldsymbol{u}(t)|^2-|\nabla\mu(t)|_{L^2}^2+\frac{1}{\kappa\rho}\|\boldsymbol{g}\|_{V^*}^2+(\eta^2C_\Omega^2+2\varepsilon c_f')|\overline{\varphi}(t)|_{L^2}^2-$$
$$\left(\eta-\frac{\varepsilon}{2}\right)\sum_{|\alpha|=k}a_\alpha|D^\alpha\overline{\varphi}|_{L^2}^2-2(\eta-\varepsilon)c_*(|f(\varphi(t))|,1+|\varphi(t)|)_{L^2}+2\varepsilon c_{f,M_0}''+\varepsilon C_\Psi\leqslant$$
$$-\frac{(\rho-\varepsilon\eta c)}{\kappa}\|\boldsymbol{u}(t)\|^2-|\nabla\mu(t)|_{L^2}^2+\frac{1}{\kappa\rho}\|\boldsymbol{g}\|_{V^*}^2-\frac{1}{2}\left(\eta-\frac{\varepsilon}{2}\right)\sum_{|\alpha|=k}a_\alpha|D^\alpha\overline{\varphi}|^2-$$
$$2(\eta-\varepsilon)c_*(|f(\varphi(t))|,1+|\varphi(t)|)_{L^2}+2\varepsilon c_{f,M_0}''+\varepsilon C_\Psi+C(\eta^2C_\Omega^2+2\varepsilon c_f')^2。$$

结合式(13)可得：

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)+\varepsilon\Psi(t)+c_2'\Big(\|\boldsymbol{u}(t)\|^2+\sum_{|\alpha|=k}a_\alpha|D^\alpha\overline{\varphi}|_{L^2}^2\Big)+2|\nabla\mu(t)|_{L^2}^2+c_3'(|f(\varphi(t))|,1+|\varphi(t)|)\leqslant$$
$$\frac{1}{\rho\kappa}\|\boldsymbol{g}\|_{V^*}^2+c_1'。\tag{18}$$

根据 Gronwall 不等式[20]可得：

$$\Psi(t)\leqslant 2\Psi(0)\mathrm{e}^{-\varepsilon t}+2\varepsilon^{-1}\left(\frac{1}{\rho K}\|\boldsymbol{g}\|_{V^*}^2+c_1'\right),\forall t\geqslant 0。\tag{19}$$

对于函数 $\Psi(t)$，存在单调不减函数 Q，使得：$\|(\boldsymbol{u}(t),\varphi(t))\|_{Y_M}^2-\langle\varphi(t)\rangle^2\leqslant\Psi(t)\leqslant Q(\|(\boldsymbol{u}(t),\varphi(t))\|_{Y_M}^2)$，所以，$\|(\boldsymbol{u}(t),\varphi(t))\|_{Y_M}^2\leqslant\Psi(t)+M_0^2\leqslant 2Q(\|(\boldsymbol{u}(0),\varphi(0))\|_{Y_M}^2)\mathrm{e}^{-\varepsilon t}+2\varepsilon^{-1}\left(\frac{1}{\rho K}\|\boldsymbol{g}\|_{V^*}^2+c_1'\right)+M_0^2$。

故存在时间 t_0 和常数 $C^2=4\varepsilon^{-1}\left(\frac{1}{\rho K}\|\boldsymbol{g}\|_{V^*}^2+c_1'\right)+2M_0^2$，当 $t\geqslant t_0$ 时，$\|(\boldsymbol{u}(t),\varphi(t))\|_{Y_M}\leqslant C$。　　证毕

定理 4　若 $k\in\mathbf{N}$，$k\geqslant 2$，$\boldsymbol{g}\in V^*$，f 满足假设 1，$(\boldsymbol{u},\varphi)$ 为方程(1)～(6)的弱解，则有不等式：

$$\int_t^{t+1}\Big(\|\boldsymbol{u}(s)\|^2+\sum_{|\alpha|=k}a_\alpha^2|D^{2\alpha}\varphi(s)|_{L^2}^2+|\mu(s)|_{H^1}^2\Big)\mathrm{d}s\leqslant Q(\|(u(0),\varphi(0))\|_{Y_M}^2)\mathrm{e}^{-\delta t}+c_0,\ \forall t\geqslant 0$$

成立。式中：Q 是单调不减函数，δ 和 c_0 是与初始值和时间无关的常数。

证明　对式(18)在$(t,t+1)$上积分可得，$\forall t\geqslant 0$，

$$\int_t^{t+1}\Big(\|\boldsymbol{u}(t)\|^2+\sum_{|\alpha|=k}a_\alpha|D^\alpha\overline{\varphi}|_{L^2}^2+|\nabla\mu(t)|_{L^2}^2+(|f(\varphi(t))|,1+|\varphi(t)|)\Big)\mathrm{d}s\leqslant$$
$$\Psi(t)+\left(\frac{1}{\rho K}\|\boldsymbol{g}\|_{V^*}^2+c_1'\right),$$

结合式(19)：

$$\int_t^{t+1}\Big(\|\boldsymbol{u}(t)\|^2+\sum_{|\alpha|=k}a_\alpha|D^\alpha\overline{\varphi}|_{L^2}^2+|\nabla\mu(t)|_{L^2}^2+(|f(\varphi(t))|,1+|\varphi(t)|)\Big)\mathrm{d}s\leqslant$$
$$2\Psi(0)\mathrm{e}^{-\varepsilon t}+(2\varepsilon^{-1}+1)\left(\frac{1}{\rho K}\|\boldsymbol{g}\|_{V^*}^2+c_1'\right)\leqslant Q(\|(\boldsymbol{u}(0),\varphi(0))\|_{Y_M}^2)\mathrm{e}^{-\varepsilon t}+c_0,\ \forall t\geqslant 0。\tag{20}$$

由式(4)、(6)，假设 1 和 Sobolev 嵌入不等式[19]可知：

$$\langle\mu(t)\rangle^2=\langle f(\varphi)\rangle^2\leqslant c_f(1+|\varphi(t)|_{L^{2m+2}}^{2m+2})\leqslant c[1+(|\nabla\varphi(t)|_{L^2}^2+\langle\varphi(t)\rangle^2)^{m+1}]。$$

故有：

$$\int_t^{t+1}\langle\mu(s)\rangle^2\mathrm{d}s\leqslant c\int_t^{t+1}[1+(|\nabla\varphi(s)|_{L^2}^2+\langle\varphi(s)\rangle^2)^{m+1}]\mathrm{d}s\leqslant Q(\|(\boldsymbol{u}(0),\varphi(0))\|_{Y_M}^2)\mathrm{e}^{-(m+1)\varepsilon t}+c_0,$$

从而：

$$\int_t^{t+1}|\mu(s)|_{H^1}^2\mathrm{d}s\leqslant Q(\|(\boldsymbol{u}(0),\varphi(0))\|_{Y_M}^2)\mathrm{e}^{-\delta t}+c_0。\tag{21}$$

由式(5)、(20)，假设 1 和 Nirenberg-Gagliardo 不等式可得：

$$\int_t^{t+1}\Big(\Big|\sum_{|\alpha|=k}a_\alpha D^{2\alpha}\varphi(s)\Big|_{L^2}^2\Big)\mathrm{d}s\leqslant$$
$$c\left[\int_t^{t+1}|\mu(s)|_{L^2}^2\mathrm{d}s+\int_t^{t+1}|f(\varphi)|_{L^2}^2\mathrm{d}s+\int_t^{t+1}\Big(\Big|\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha D^{2\alpha}\varphi(s)\Big|_{L^2}^2\Big)\mathrm{d}s\right]\leqslant$$
$$c\left[\int_t^{t+1}|\mu(s)|_{L^2}^2\mathrm{d}s+\int_t^{t+1}|f(\varphi)|_{L^2}^2\mathrm{d}s+\int_t^{t+1}\Big(\sum_{i=1}^{k-1}\sum_{|\alpha|=i}|a_\alpha||D^{2\alpha}\varphi(s)|_{L^2}^2\Big)\mathrm{d}s\right]\leqslant$$
$$c\left[\int_t^{t+1}|\mu(s)|_{L^2}^2\mathrm{d}s+\int_t^{t+1}|f(\varphi)|_{L^2}^2\mathrm{d}s+\int_t^{t+1}(\varepsilon|\varphi(s)|_{H^{2k}}^2+c_\varepsilon|\varphi(s)|_{L^2}^2)\mathrm{d}s\right]。$$

从而有：

$$\int_t^{t+1}\Big(\sum_{|\alpha|=k}a_\alpha^2|D^{2\alpha}\varphi(s)|_{L^2}^2\Big)\mathrm{d}s\leqslant c\left[\int_t^{t+1}|\mu(s)|_{L^2}^2\mathrm{d}s+\int_t^{t+1}|f(\varphi)|_{L^2}^2\mathrm{d}s+\int_t^{t+1}|\varphi(s)|_{L^2}^2\mathrm{d}s\right]\leqslant$$
$$Q(\|(\boldsymbol{u}(0),\varphi(0))\|_{Y_M}^2)\mathrm{e}^{-\delta t}+c_1。\tag{22}$$

于是由式(20)～(22)可以完成该定理的证明。　　证毕

定理 5　若 $k\in\mathbf{N}$，$k\geqslant 2$，$\boldsymbol{g}\in H$，f 满足假设 1，则 $S(t)$ 在 $V\times H^{k+1}$ 存在有界吸收集，即对 $\forall\Sigma_1\subset Y_M$，存在 $t_1=t_1(\Sigma_1)$，使得 $\|S(t)(u_0,\varphi_0)\|_{V\times H^{k+1}}\leqslant C$，$\forall t\geqslant t_1$。

证明　在式(7)中令 $\boldsymbol{v}=-2\Delta\boldsymbol{u}$ 可得：

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2+\rho|\Delta\boldsymbol{u}|^2=((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\Delta\boldsymbol{u})-2\kappa(\mu\nabla\varphi,\Delta\boldsymbol{u})-2(\boldsymbol{g},\Delta\boldsymbol{u})。\tag{23}$$

令 $\overline{\varphi}(t)=\varphi(t)-M_0$，$\overline{\mu}(t)=\mu(t)-\langle\mu(t)\rangle$，分别用 $\sum_{|\alpha|=k}(-1)^{k+1}a_\alpha\Delta D^{2\alpha}\overline{\varphi}$ 和 $2\Delta^2\overline{\mu}-2\eta\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta^2D^{2\alpha}\overline{\varphi}$ 与式(4)、(5)作 L^2 内积得：

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|_{L^2}^2+2|\Delta\overline{\mu}|^2+2\eta\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2=$$
$$2((\boldsymbol{u}\cdot\nabla\overline{\varphi}),\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi})-2\eta(\Delta\overline{\mu},\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi})+$$
$$2(\Delta\overline{\mu},\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha(-1)^i\Delta D^{2\alpha}\overline{\varphi})-2\eta(\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha(-1)^i\Delta D^{2\alpha}\overline{\varphi},\sum_{|\alpha|=k}(-1)^k a_\alpha\Delta D^{2\alpha}\overline{\varphi})+$$
$$2(\Delta(f(\varphi)-\langle f(\varphi)\rangle),\Delta\overline{\mu}-\eta\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi})。\tag{24}$$

由式(23)、(24)可得:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{u}\|^2+\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|_{L^2}^2)+2\rho|\Delta\boldsymbol{u}|^2+2|\Delta\overline{\mu}|^2+2\eta\sum_{|\alpha|=k}a_\alpha|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2=$$
$$2((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\Delta\boldsymbol{u})-2\kappa(\mu\nabla\varphi,\Delta\boldsymbol{u})-2(\boldsymbol{g},\Delta\boldsymbol{u})+2((\boldsymbol{u}\cdot\nabla\overline{\varphi}),\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi})-$$
$$2\eta(\Delta\overline{\mu},\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi})+2(\Delta\overline{\mu},\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha(-1)^i\Delta D^{2\alpha}\overline{\varphi})-$$
$$2\eta(\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha(-1)^i\Delta D^{2\alpha}\overline{\varphi},\sum_{|\alpha|=k}(-1)^k a_\alpha\Delta D^{2\alpha}\overline{\varphi})+2(\Delta(f(\varphi)-\langle f(\varphi)\rangle),\Delta\overline{\mu}-\eta\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi})。\tag{25}$$

下面将估计式(25)右边各项。

$$2|((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\Delta\boldsymbol{u})|\leqslant 2|\boldsymbol{u}|^{\frac12}\|\boldsymbol{u}\||\Delta\boldsymbol{u}|^{\frac32}\leqslant C|\boldsymbol{u}|^2\|\boldsymbol{u}\|^4+\frac{\rho}{2}|\Delta\boldsymbol{u}|^2,$$

$$2|\kappa(\mu\nabla\varphi,\Delta\boldsymbol{u})|\leqslant 2\kappa|\mu\nabla\varphi|_{L^2}|\Delta\boldsymbol{u}|\leqslant 2\kappa|\mu|_{L^\infty}|\nabla\varphi|_{L^2}|\Delta\boldsymbol{u}|\leqslant C|\mu|_{L^2}^{\frac12}|\Delta\mu|_{L^2}^{\frac12}|\nabla\varphi|_{L^2}|\Delta\boldsymbol{u}|\leqslant$$
$$C|\mu|_{L^2}^{\frac23}|\nabla\varphi|_{L^2}^{\frac43}|\Delta\boldsymbol{u}|^{\frac43}+\frac14|\Delta\mu|_{L^2}^2\leqslant C|\mu|_{L^2}^2|\nabla\varphi|_{L^2}^4+\frac{\rho}{2}|\Delta\boldsymbol{u}|^2+\frac14|\Delta\mu|_{L^2}^2\leqslant$$
$$C|\mu|_{L^2}^2|\nabla\overline{\varphi}|_{L^2}^2\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|_{L^2}^2+\frac{\rho}{2}|\Delta\boldsymbol{u}|^2+\frac14|\Delta\overline{\mu}|_{L^2}^2。$$

$$2|(\boldsymbol{g},\Delta\boldsymbol{u})|\leqslant\frac{\rho}{2}|\Delta\boldsymbol{u}|^2+\rho^{-1}|\boldsymbol{g}|^2,$$

$$2\left|((\boldsymbol{u}\cdot\nabla\overline{\varphi}),\sum_{|\alpha|=k}(-1)^k a_\alpha\Delta D^{2\alpha}\overline{\varphi})\right|\leqslant 2|\boldsymbol{u}\cdot\nabla\overline{\varphi}|_{L^2}\left|\sum_{|\alpha|=k}(-1)^k a_\alpha\Delta D^{2\alpha}\overline{\varphi})\right|_{L^2}\leqslant$$
$$C|\nabla\overline{\varphi}|_{L^2}|\nabla\Delta\overline{\varphi}|_{L^2}|\boldsymbol{u}|\|\boldsymbol{u}\|+\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|^2\leqslant C(|\nabla\overline{\varphi}|_{L^2}^2|\nabla\Delta\overline{\varphi}|_{L^2}^2+|\boldsymbol{u}|^2\|\boldsymbol{u}\|^2)+$$
$$\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2\leqslant C\left(|\nabla\overline{\varphi}|_{L^2}^2\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|_{L^2}^2+|\boldsymbol{u}|^2\|\boldsymbol{u}\|^2\right)+\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2。$$

$$2\eta\left|(\Delta\overline{\mu},\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi})\right|\leqslant 2\eta|\Delta\overline{\mu}|_{L^2}\left|\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi}\right|_{L^2}\leqslant$$
$$2\eta(c|\Delta\overline{\mu}|_{L^2}^2+\frac{1}{10}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|^2)\leqslant 2\eta c|\Delta\overline{\mu}|_{L^2}^2+\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2。$$

$$2\left|(\Delta\overline{\mu},\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha(-1)^i D^{2\alpha}\overline{\varphi})\right|\leqslant 2|\Delta\overline{\mu}|_{L^2}\left|\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha(-1)^i D^{2\alpha}\overline{\varphi}\right|_{L^2}\leqslant$$
$$\frac14|\Delta\overline{\mu}|_{L^2}^2+C\left|\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha(-1)^i D^{2\alpha}\overline{\varphi}\right|_{L^2}^2\leqslant\frac14|\Delta\overline{\mu}|_{L^2}^2+C\sum_{i=1}^{k-1}\sum_{|\alpha|=k}a_\alpha|\Delta D^{2\alpha}\overline{\varphi}|^{\frac{2i}{2k+2}}|\overline{\varphi}|_{L^2}^{1-\frac{2i}{2k+2}}\leqslant$$
$$\frac14|\Delta\overline{\mu}|_{L^2}^2+\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2+c_1|\overline{\varphi}|_{L^2}^2+C_1。$$

$$2\eta\left|(\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha(-1)^i D^{2\alpha}\overline{\varphi},\sum_{|\alpha|=k}(-1)^k a_\alpha D^{2\alpha}\overline{\varphi})\right|_{L^2}\leqslant\eta c\sum_{i=1}^{k-1}\sum_{|\alpha|=i}a_\alpha|D^{2\alpha}\overline{\varphi}|_{L^2}^2+\frac{\eta}{10}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2)\leqslant$$
$$\eta c\sum_{i=1}^{k-1}\sum_{|\alpha|=k}a_\alpha|\Delta D^{2\alpha}\overline{\varphi}|^{\frac{2i}{2k+2}}|\overline{\varphi}|_{L^2}^{1-\frac{2i}{2k+2}}+\frac{\eta}{10}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2\leqslant$$
$$\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2+c_2|\overline{\varphi}|_{L^2}^2+C_2。$$

$$
\begin{aligned}
&2|(\Delta(f(\varphi)-\langle f(\varphi)\rangle),\Delta\overline{\mu})| = 2|(\Delta f(\varphi),\Delta\overline{\mu})|\leqslant 2|\Delta f(\varphi)|_{L^2}|\Delta\overline{\mu}|_{L^2}\leqslant\frac{1}{4}|\Delta\overline{\mu}|_{L^2}^2+c|\Delta f(\varphi)|_{L^2}^2\leqslant\\
&\frac{1}{4}|\Delta\overline{\mu}|_{L^2}^2+c(|f'(\varphi)\Delta\overline{\varphi}|_{L^2}^2+|f''(\varphi)|\nabla\varphi|^2|_{L^2}^2)\leqslant\frac{1}{4}|\Delta\overline{\mu}|_{L^2}^2+|f'(\varphi)\Delta\overline{\varphi}|_{L^2}^2+|f''(\varphi)|\nabla\varphi|^2|_{L^2}^2\leqslant\\
&\frac{1}{4}|\Delta\overline{\mu}|_{L^2}^2+|f'(\varphi)\Delta\overline{\varphi}|_{L^2}^2+|f''(\varphi)|\nabla\varphi|^2|_{L^2}^2\leqslant\frac{1}{4}|\Delta\overline{\mu}|_{L^2}^2+cQ(|\varphi|_{H^1})(|\Delta\overline{\varphi}|_{L^2}^2+|\nabla\varphi|_{L^4}^4)\leqslant\\
&\frac{1}{4}|\Delta\overline{\mu}|_{L^2}^2+cQ(|\varphi|_{H^1})\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|^2。
\end{aligned}
$$

$$
\begin{aligned}
&2\left|(\Delta(f(\varphi)-\langle f(\varphi)\rangle),\eta\sum_{|\alpha|=k}a_\alpha(-1)^k\Delta D^{2\alpha}\overline{\varphi})\right|_{L^2}\leqslant\\
&\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2+c(|f'(\varphi)\Delta\overline{\varphi}|_{L^2}^2+|f''(\varphi)|\nabla\varphi|^2|_{L^2}^2)\leqslant\\
&\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2+cQ(|\varphi|_{H^1})(|\Delta\overline{\varphi}|_{L^2}^2+|\nabla\varphi|_{L^4}^4)\leqslant\\
&\frac{\eta}{5}\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|_{L^2}^2+cQ(|\varphi|_{H^1})\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|^2。
\end{aligned}
$$

将以上估计式代入式(25)可得：

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{u}\|^2+\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|^2)+\frac{\rho}{2}|\Delta\boldsymbol{u}|^2+\left(\frac{5}{4}-2\eta c\right)|\Delta\overline{\mu}|^2+\eta\sum_{|\alpha|=k}a_\alpha^2|\Delta D^{2\alpha}\overline{\varphi}|^2\leqslant\\
&C|\boldsymbol{u}|^2\|\boldsymbol{u}\|^4+C|\mu|_{L^2}^2|\nabla\overline{\varphi}|_{L^2}^2\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|^2+C\left(|\nabla\overline{\varphi}|_{L^2}^2\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|^2+|\boldsymbol{u}|^2\|\boldsymbol{u}\|^2\right)+\\
&c_3|\overline{\varphi}|_{L^2}^2+C_3+cQ(|\varphi|_{H^1})\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|^2+\rho^{-1}|\boldsymbol{g}|^2。
\end{aligned}
\tag{26}
$$

取 $\eta=\dfrac{1}{8c}$，令 $K_1(t)=C(|\boldsymbol{u}|^2\|\boldsymbol{u}\|^2+|\mu|_{L^2}^2|\nabla\overline{\varphi}|_{L^2}^2+|\nabla\overline{\varphi}|_{L^2}^2+|\boldsymbol{u}|^2+Q(|\varphi|_{H^1}))$，$K_2(t)=\rho^{-1}|\boldsymbol{g}|^2+c_3|\overline{\varphi}|_{L^2}^2+C_3$，$\Upsilon(t)=\|\boldsymbol{u}\|^2+\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|^2$，则式(26)可写成：$\dfrac{\mathrm{d}}{\mathrm{d}t}\Upsilon(t)\leqslant K_1(t)\Upsilon(t)+K_2(t)$。

由定理 3 和定理 4 可知，当 $t\geqslant t_0$ 时，有不等式：$\int_t^{t+1}K_1(s)\mathrm{d}s\leqslant b_1$，$\int_t^{t+1}K_2(s)\mathrm{d}s\leqslant b_1$，$\int_t^{t+1}\Upsilon(t)\mathrm{d}s\leqslant b_3$ 成立。

根据一致 Gronwall 不等式[19,21]可以得到：$\Upsilon(t+1)\leqslant(b_1+b_2)\mathrm{e}^{b_3}$，当 $t\geqslant t_0$ 时。

所以，当 $t\geqslant t_1=t_0+1$ 时，$\|\boldsymbol{u}\|^2+\sum_{|\alpha|=k}a_\alpha|\nabla D^\alpha\overline{\varphi}|^2\leqslant c^2=(b_1+b_2)\mathrm{e}^{b_3}$ 成立，即：

$$\|S(t)(u_0,\varphi_0)\|_{V\times H^{k+1}}\leqslant C。$$

证毕

定理 6　若 $k\in\mathbf{N}$，$k\geqslant 2$，$\boldsymbol{g}\in H$，f 满足假设 1，则 $S(t)$ 在 Y_M 中整体吸引子 A_M。

证明　由定理 3 和定理 5 可得 $S(t)$ 在 Y_M 和 $V\times H^{k+1}$ 中存在有界吸收集，由 $V\times H^{k+1}$ 紧嵌入 Y_M 可得 $S(t)$ 在 Y_M 中的一致紧性，根据吸引子存在性定理[21]可得该定理成立。　证毕

参考文献：

[1] GAL C G, GRASSELLI M. Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D[J]. Annales de l'Institut Henri Poincare Analysis Non Lineaire, 2010, 27(1): 401-436.

[2] GAL C G, GRASSELLI M, MIRANVILLE A. Cahn-Hilliard-Navier-Stokes systems with moving contact lines[J]. Calculus of Variations Partial Differential Equations, 2016, 55(3): 1-47.

[3] YOU B. Global attractor of the Cahn-Hilliard-Navier-Stokes system with moving contact lines[J]. Communications on Pure and Applied Analysis, 2019, 18(5): 2283-2298.

[4] ZHOU Y, FAN J S. The vanishing viscosity limit for a 2D Cahn-Hilliard-Navier-Stokes system with a slip boundary condition[J]. Nonlinear Analysis: Real World Applications, 2013, 14(2): 1130-1134.

[5] GIORGINI A, MIRANVILLE A, TEMAM R. Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system[J]. SIAM Journal on Mathematical Analysis, 2019, 51(3): 2535-2574.

[6] YOU B. Pullback exponential attractors for the viscous Cahn-Hilliard-Navier-Stokes system with dynamic boundary conditions[J]. Journal of Mathematical Analysis and Applications, 2018, 478(2): 321-344.

[7] YOU B, LI F, ZHANG C. Finite dimensional global attractor of the Cahn-Hilliard-Navier-Stokes system with dynamic boundary conditions[J]. Communications in Mathematical Sciences, 2018, 16(1): 53-76.
[8] HOSSEINI B S, TUREK S, MOLLER M, et al. Isogeometric analysis of the Navier-Stokes-Cahn-Hilliard equations with application to incompressible two-phase flows[J]. Journal of Computational Physics, 2016, 348(1): 171-194.
[9] CHERFILS L, PETCU M. Energy stable numerical scheme for the viscous Cahn-Hilliard-Navier-Stokes equations with moving contact line[J]. Numerical Methods for Partial Differential Equations, 2019, 35(3): 1113-1133.
[10] HAN D Z, BRYLEV A, YANG X F. Numerical analysis of second order, fully discrete energy stable schemes for phase field models of two-phase incompressible flows[J]. Journal of Scientific Computing, 2017, 70(3): 965-989.
[11] CHEN L, ZHAO J. A novel second-order linear scheme for the Cahn-Hilliard-Navier-Stokes equations[J]. Journal of Computational Physics, 2020, 423: 109782.
[12] CHERFILS L, GATTI S, MIRANVILLE A. Asymptotic behavior of higher-order Navier-Stokes-Cahn-Hilliard systems[J]. Mathematical Methods in the Applied Sciences, 2018, 41(12): 4476-4794.
[13] ESENTURK E, CAGINALP G. Anisotropic phase field equations of arbitrary order[J]. Discrete and Continuous Dynamical Systems-Series S, 2010, 4(2): 311-350.
[14] CHERFILS L, MIRANVILLE A, PENG S R. Higher-order anisotropic models in phase separation[J]. Advances in Nonlinear Analysis, 2019, 8(1): 278-302.
[15] CHERFILS L, MIRANVILLE A, PENG S R, et al. Higher-order generalized Cahn-Hilliard equations[J]. Electronic Journal of Qualitative Theory of Differential Equations, 2017, 2017(9): 1-22.
[16] PENG S R, ZHU H Y. Well-posedness for modified higher-order anisotropic Cahn-Hilliard equations[J]. Asymptotic Analysis, 2019, 111(3/4): 201-215.
[17] 罗娇,漆前,罗宏. 高阶各向异性 Cahn-Hilliard-Navier-Stokes 系统的弱解[J]. 数学物理学报, 2020, 40A(6): 1599-1611.
LUO J, QI Q, LUO H. Weak solutions to higher-order anisotropic Cahn-Hilliard-Navier-Stokes systems[J]. Acta Mathematica Scientia, 2020, 40A(6): 1599-1611.
[18] 罗娇. 高阶各向异性 Cahn-Hilliard-Navier-Stokes 系统的吸引子[D]. 成都:四川师范大学, 2021.
LUO J. Attractors of higher-order anisotropic Cahn-Hilliard-Navier-Stokes Systems[D]. Chengdu: Sichuan Normal University, 2021.
[19] 马天. 偏微分方程理论和方法[M]. 北京:科学出版社, 2011.
MA T. Theory and method of partial differential equations[M]. Beijing: Science Press, 2011.
[20] AGMON S, DOUGLIS A, NIRENBERG L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II[J]. Communications on Pure and Applied Mathematics, 1964, 17(1): 35-92.
[21] TEMAM R. Infinite-dimensional dynamical systems in mechanics and physics[M]. New York: Springer-Verlag, 1997.

Large Time Behavior of Cahn-Hilliard-Navier-Stokes Equations

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Abstract: [**Purposes**]The Cahn-Hilliard-Navier-Stokes equations, which are coupled by incompressible Navier-Stokes equations and higher-order anisotropic Cahn-Hilliard equations, are widely used and play an important role. [**Methods**]Bounded absorbing sets and uniform compactness of solution semigroup to the equations are obtained by using energy estimates. [**Findings**]Then the existence of attractors are proved according to the existence theorem of attractor. [**Conclusions**]The large time behavior of the equations under Neumann boundary condition for relative concentrations is studied.

Keywords: Cahn-Hilliard-Navier-Stokes equations; Neumann boundary condition; absorbing set; global attractor

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