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多项时间-两边空间分数阶对流-扩散方程的加权隐式数值解*

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摘要:考虑多项时间-两边空间分数阶对流-扩散方程的初边值问题,基于移位 Grünwald-Letnikov 公式,将方程中的空间分数阶导数采用加权平均有限差分法近似,得到一种加权隐式有限差分格式。利用能量估计,得到了该差分格式的稳定性。然后利用数学归纳法证明了在相同的条件下,所提出的差分格式是收敛的。最后通过数值例子说明了所提出的差分格式是可靠和有效的,并对方程的数值解和精确解进行了比较,验证了本文的理论结果。

关键词:分数阶对流-扩散方程;空间分数阶导数;加权隐式格式;收敛性;稳定性;有限差分法

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有限差分方法是数值求解微分方程的常用方法之一。有限差分法主要是将连续的微分问题变成离散的代数问题进行数值逼近,即将求解的区域进行网格剖分,用离散的有限节点来代替连续的求解区域,然后用泰勒展开等方法,把微分方程中的连续微分项用离散的差商来代替,最后形成基于离散网格点的一个以节点值为未知项的离散线性方程组[1-6]。

相较整数阶微分方程,分数阶微分方程具有遗传记忆效应,能更好地模拟自然物理过程和动力系统过程,因此近年来分数阶微分方程在流体力学、材料力学、生物学、等离子体物理学、金融学、化学、水文地理等领域得到了广泛应用[1-14]。分数阶对流-扩散方程能更精确地模拟具有长尾性态的溶质运动过程,目前许多学者致力于这一类分数阶微分方程的研究。绝大部分分数阶对流-扩散方程的初边值问题不存在解析解,但随着计算机技术的发展及数值分析领域的突起,使得用数值方法逼近解决分数阶对流-扩散方程的初边值问题成为现实。借助计算机进行分数阶对流-扩散方程的计算和模拟,能够恰当地模拟和分析该方程的长尾性态等各种性质。因此,越来越多的学者开始研究如何利用有限差分方法对分数阶对流-扩散方程进行数值模拟,例如,Momani 和 Yang 等人[15-16]研究了反常对流-扩散方程在地质水文中的应用,即在多孔介质中可用分数阶导数来模拟由流体流动引起的被动示踪运输问题;Liu 等人[17]考虑了时间分数阶对流-扩散方程,利用 Melli 变换和拉普拉斯变换得到了此方程的精确解;马亮亮等人[18]给出了变系数空间分数阶对流-扩散方程的数值解法;顾先明等人[19]给出了时间空间分数阶对流-扩散方程的二阶隐式差分格式快速迭代方法。

本文考虑多项时间-两边空间分数阶对流-扩散方程的初边值问题:

$$\sum_{l=1}^{k}\theta_l\frac{\partial^{\gamma_l}u(x,t)}{\partial t^{\gamma_l}}=-v(x,t)\frac{\partial u(x,t)}{\partial x}+a_+(x,t)\frac{\partial^{\alpha}u(x,t)}{\partial_+ x^{\alpha}}+a_-(x,t)\frac{\partial^{\alpha}u(x,t)}{\partial_- x^{\alpha}}+$$

$$b_+(x,t)\frac{\partial^{\beta}u(x,t)}{\partial_+ x^{\beta}}+b_-(x,t)\frac{\partial^{\beta}u(x,t)}{\partial_- x^{\beta}}+f(x,t),0\leqslant x\leqslant L,0\leqslant t\leqslant T;\tag{1}$$

$$u(x,0)=g(x),0\leqslant x\leqslant L;\tag{2}$$

$$u(0,t)=0,u(L,t)=\varphi(t),0\leqslant t\leqslant T;\tag{3}$$

式中:$0<\gamma_1,\gamma_2,\cdots,\gamma_k<1$,$1<\alpha,\beta<2$,$\theta_1,\theta_2,\cdots,\theta_k\geqslant0$,$\theta_1+\theta_2+\cdots+\theta_k=1$,$v(x,t)\geqslant0$,$a_+(x,t)\geqslant0$,$a_-(x,t)\geqslant0$,$b_+(x,t)\geqslant0$,$b_-(x,t)\geqslant0$ 是$[0,L]\times[0,T]$上的连续函数;$\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}}$是 Caputo 分数阶导数[1]:

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$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}=\frac{1}{\Gamma(1-\gamma)}\int_0^t (t-\eta)^{-\gamma}\frac{\partial u(x,\eta)}{\partial \eta}\mathrm{d}\eta,0<\gamma<1;$$

式中:$\Gamma(\cdot)$是伽马函数;$\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}$,$\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}$是 Riemann-Liouville 分数阶导数[1]:

$$\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}=\frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_0^x \frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}}\mathrm{d}\xi,\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_x^L \frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}}\mathrm{d}\xi;$$

式中:$0\leqslant n-1<\alpha<n$(n 是整数)。

关于此类问题的数值解法,前人已经做了一些研究。Meerschaert 等人[20-21]分别对单边和双边对流-扩散方程利用改进型 Grünwald-LetNikov 差分方法进行求解,但结果表明用 Grünwald 得到的显式差分格式是不稳定的,因此数值解不会收敛于方程的精确解。夏源等人[22]对单边的时间和空间分数阶对流-弥散方程给出了类似的算法。Liu 等人[23]用此类算法对空间分数阶福克-普朗克方程进行了离散计算。马维元等人[24]对空间-时间分数阶扩散方程的初边值问题提出了一种加权平均差分格式。张红玉等人[25]给出了一类空间分数阶对流-扩散方程的加权平均有限差分解法。苏丽娟等人[26]给出了双边空间分数阶对流-扩散方程的一种隐式有限差分解法。汪向艳等人[27]给出了变系数空间分数阶扩散方程的一种加权显式有限差分方法。本文在上述文献的研究基础上,进一步研究多项时间-两边空间分数阶对流-扩散方程的一种加权隐式差分解法。

1 加权有限差分格式

为了数值求解方程(1)~(3),首先对求解的区域进行网格剖分。考虑区域$[0,L]\times[0,T]$,给定等距剖分,选取正整数 M,N,并令 $h=L/M,\tau=T/N$。记 $x_i=ih(i=0,1,2,\cdots,M)$;$t_n=n\tau(n=0,1,2,\cdots,N)$;$0=x_0<x_1<\cdots<x_M=L$;$0=t_0<t_1<\cdots<t_N=T$;空间步长 $h=x_{i+1}-x_i(i=0,1,2,\cdots,M-1)$,时间步长 $\tau=t_{n+1}-t_n(n=0,1,2,\cdots,N-1)$;网格点为$(x_i,t_n)$。为简便起见,文中假设 $\gamma_1=\max\{\gamma_1,\gamma_2,\cdots,\gamma_k\}$。

对于多项时间分数阶导数$\frac{\partial^{\gamma_1}u(x,t)}{\partial t^{\gamma_1}}$,$\frac{\partial^{\gamma_2}u(x,t)}{\partial t^{\gamma_2}}$,…,$\frac{\partial^{\gamma_k}u(x,t)}{\partial t^{\gamma_k}}$采用下列有限差分近似离散:

$$\begin{cases}\frac{\partial^{\gamma_1}u(x_i,t_n)}{\partial t^{\gamma_1}}=\frac{\tau^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\sum_{j=0}^{n}\frac{u(x_i,t_{n+1-j})-u(x_i,t_{n-j})}{\tau}[(j+1)^{1-\gamma_1}-j^{1-\gamma_1}]+O(\tau),\\ \frac{\partial^{\gamma_2}u(x_i,t_n)}{\partial t^{\gamma_2}}=\frac{\tau^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\sum_{j=0}^{n}\frac{u(x_i,t_{n+1-j})-u(x_i,t_{n-j})}{\tau}[(j+1)^{1-\gamma_2}-j^{1-\gamma_2}]+O(\tau),\\ \cdots\cdots\\ \frac{\partial^{\gamma_k}u(x_i,t_n)}{\partial t^{\gamma_k}}=\frac{\tau^{1-\gamma_k}}{\Gamma(2-\gamma_k)}\sum_{j=0}^{n}\frac{u(x_i,t_{n+1-j})-u(x_i,t_{n-j})}{\tau}[(j+1)^{1-\gamma_k}-j^{1-\gamma_k}]+O(\tau)。\end{cases}\tag{4}$$

对一阶空间分数阶导数$\frac{\partial u(x,t)}{\partial x}$采用向后差商近似:

$$\frac{\partial u(x_i,t_n)}{\partial x}=\frac{u(x_i,t_n)-u(x_{i-1},t_n)}{h}+O(h)。\tag{5}$$

借助于移位 Grünwald-Letnikov 公式,Riemann-Liouville 型两边空间分数阶导数$\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}$,$\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}$,$\frac{\partial^\beta u(x,t)}{\partial_+ x^\beta}$,$\frac{\partial^\beta u(x,t)}{\partial_- x^\beta}$可分别离散为:

$$\frac{\partial^\alpha u(x_i,t_n)}{\partial_+ x^\alpha}=\frac{1}{h^\alpha}\sum_{j=0}^{i+1}g_j^{(\alpha)}u(x_i-(j-1)h,t_n)+O(h),\tag{6}$$

$$\frac{\partial^\alpha u(x_i,t_n)}{\partial_- x^\alpha}=\frac{1}{h^\alpha}\sum_{j=0}^{M-i+1}g_j^{(\alpha)}u(x_i+(j-1)h,t_n)+O(h),\tag{7}$$

$$\frac{\partial^\beta u(x_i,t_n)}{\partial_+ x^\beta}=\frac{1}{h^\beta}\sum_{j=0}^{i+1}g_j^{(\beta)}u(x_i-(j-1)h,t_n)+O(h),\tag{8}$$

$$\frac{\partial^{\beta}u(x_i,t_n)}{\partial_- x^{\beta}}=\frac{1}{h^{\beta}}\sum_{j=0}^{M-i+1}g_j^{(\beta)}u(x_i+(j-1)h,t_n)+O(h)。\tag{9}$$

式中：$g_0^{(\alpha)}=1,g_j^{(\alpha)}=\left(1-\frac{\alpha+1}{j}\right)g_{j-1}^{(\alpha)},j=1,2,3,\cdots;g_0^{(\beta)}=1,g_j^{(\beta)}=\left(1-\frac{\beta+1}{j}\right)g_{j-1}^{(\beta)},j=1,2,3,\cdots$。

令 u_i^n 为微分方程(1)～(3)的解的近似值，$v_i^n=v(x_i,t_n)$，$a_{+i}^n=a_+(x_i,t_n)$，$a_{-i}^n=a_-(x_i,t_n)$，$b_{+i}^n=b_+(x_i,t_n)$，$b_{-i}^n=b_-(x_i,t_n)$，$f_i^n=f(x_i,t_n)$。将式(4)～(9)代入式(1)，并用加权平均近似方程(1)在(x_i,t_n)处的导数，得到如下加权隐式差分格式：

$$\theta_1\frac{\tau^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_1}-j^{1-\gamma_1}]+\theta_2\frac{\tau^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_2}-j^{1-\gamma_2}]+$$
$$\cdots+\theta_k\frac{\tau^{1-\gamma_k}}{\Gamma(2-\gamma_k)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_k}-j^{1-\gamma_k}]=-v_i^n\left[\mu\frac{u_i^n-u_{i-1}^n}{h}+(1-\mu)\frac{u_i^{n+1}-u_{i-1}^{n+1}}{h}\right]+$$
$$\frac{a_{+i}^n}{h^{\alpha}}\left[\mu\sum_{j=0}^{i+1}g_j^{(\alpha)}u_{i-j+1}^n+(1-\mu)\sum_{j=0}^{i+1}g_j^{(\alpha)}u_{i-j+1}^{n+1}\right]+\frac{a_{-i}^n}{h^{\alpha}}\left[\mu\sum_{j=0}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^n+(1-\mu)\sum_{j=0}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^{n+1}\right]+$$
$$\frac{b_{+i}^n}{h^{\beta}}\left[\mu\sum_{j=0}^{i+1}g_j^{(\beta)}u_{i-j+1}^n+(1-\mu)\sum_{j=0}^{i+1}g_j^{(\beta)}u_{i-j+1}^{n+1}\right]+\frac{b_{-i}^n}{h^{\beta}}\left[\mu\sum_{j=0}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^n+(1-\mu)\sum_{j=0}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^{n+1}\right]+$$
$$\mu f_i^n+(1-\mu)f_i^{n+1}。\tag{10}$$

式中：$i=1,2,\cdots,M-1,n=0,1,2,\cdots,N-1,\mu$ 是权参数，且 $0\leqslant\mu\leqslant1$。

1) 当 $\mu=0$ 时，式(10)变为隐式差分格式：

$$\theta_1\frac{\tau^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_1}-j^{1-\gamma_1}]+\theta_2\frac{\tau^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_2}-j^{1-\gamma_2}]+$$
$$\cdots+\theta_k\frac{\tau^{1-\gamma_k}}{\Gamma(2-\gamma_k)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_k}-j^{1-\gamma_k}]=-v_i^n\left(\frac{u_i^{n+1}-u_{i-1}^{n+1}}{h}\right)+$$
$$\frac{a_{+i}^n}{h^{\alpha}}\sum_{j=0}^{i+1}g_j^{(\alpha)}u_{i-j+1}^{n+1}+\frac{a_{-i}^n}{h^{\alpha}}\sum_{j=0}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^{n+1}+\frac{b_{+i}^n}{h^{\beta}}\sum_{j=0}^{i+1}g_j^{(\beta)}u_{i-j+1}^{n+1}+\frac{b_{-i}^n}{h^{\beta}}\sum_{j=0}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^{n+1}+f_i^{n+1}。\tag{11}$$

2) 当 $\mu=1$ 时，式(10)变成显式差分格式：

$$\theta_1\frac{\tau^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_1}-j^{1-\gamma_1}]+\theta_2\frac{\tau^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_2}-j^{1-\gamma_2}]+$$
$$\cdots+\theta_k\frac{\tau^{1-\gamma_k}}{\Gamma(2-\gamma_k)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_k}-j^{1-\gamma_k}]=-v_i^n\left(\frac{u_i^n-u_{i-1}^n}{h}\right)+\frac{a_{+i}^n}{h^{\alpha}}\sum_{j=0}^{i+1}g_j^{(\alpha)}u_{i-j+1}^{n+1}+$$
$$\frac{a_{-i}^n}{h^{\alpha}}\sum_{j=0}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^n+\frac{b_{+i}^n}{h^{\beta}}\sum_{j=0}^{i+1}g_j^{(\beta)}u_{i-j+1}^n+\frac{b_{-i}^n}{h^{\beta}}\sum_{j=0}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^n+f_i^n。\tag{12}$$

3) 当 $\mu=1/2$ 时，式(10)变成空间-时间分数阶 Crank-Nicolson 差分格式：

$$\theta_1\frac{\tau^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_1}-j^{1-\gamma_1}]+\theta_2\frac{\tau^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_2}-j^{1-\gamma_2}]+$$
$$\cdots+\theta_k\frac{\tau^{1-\gamma_k}}{\Gamma(2-\gamma_k)}\sum_{j=0}^{n}\frac{u_i^{n+1-j}-u_i^{n-j}}{\tau}[(j+1)^{1-\gamma_k}-j^{1-\gamma_k}]=-\frac{v_i^n}{2}\left(\frac{u_i^n-u_{i-1}^n}{h}+\frac{u_i^{n+1}-u_{i-1}^{n+1}}{h}\right)+$$
$$\frac{a_{+i}^n}{2h^{\alpha}}\left(\sum_{j=0}^{i+1}g_j^{(\alpha)}u_{i-j+1}^n+\sum_{j=0}^{i+1}g_j^{(\alpha)}u_{i-j+1}^{n+1}\right)+\frac{a_{-i}^n}{2h^{\alpha}}\left(\sum_{j=0}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^n+\sum_{j=0}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^{n+1}\right)+$$
$$\frac{b_{+i}^n}{2h^{\beta}}\left(\sum_{j=0}^{i+1}g_j^{(\beta)}u_{i-j+1}^n+\sum_{j=0}^{i+1}g_j^{(\beta)}u_{i-j+1}^{n+1}\right)+\frac{b_{-i}^n}{2h^{\beta}}\left(\sum_{j=0}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^n+\sum_{j=0}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^{n+1}\right)+\frac{f_i^n+f_i^{n+1}}{2}。\tag{13}$$

方程(10)也可以写成如下形式的加权隐式差分格式：

1) 当 $n=0$ 时，有：

$$-(1-\mu)(\xi_i^0+\eta_i^0g_2^{(\alpha)}+\omega_i^0g_2^{(\beta)}+\zeta_i^0+\psi_i^0)u_{i-1}^1+$$
$$[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_i^0-\eta_i^0g_1^{(\alpha)}-\omega_i^0g_1^{(\beta)}-\zeta_i^0g_1^{(\alpha)}-\psi_i^0g_1^{(\beta)})]u_i^1-$$
$$(1-\mu)(\eta_i^0+\omega_i^0+\zeta_i^0g_2^{(\alpha)}+\psi_i^0g_2^{(\beta)})u_{i+1}^1-(1-\mu)\eta_i^0\sum_{j=3}^{i+1}g_j^{(\alpha)}u_{i-j+1}^1-(1-\mu)\omega_i^0\sum_{j=3}^{i+1}g_j^{(\beta)}u_{i-j+1}^1-$$

$$(1-\mu)\zeta_i^0\sum_{j=3}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^1-(1-\mu)\psi_i^0\sum_{j=3}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^1=\mu(\xi_i^0+\eta_i^0g_2^{(\alpha)}+\omega_i^0g_2^{(\beta)}+\zeta_i^0+\psi_i^0)u_{i-1}^0+$$
$$[\delta_1+\delta_2+\cdots+\delta_k-\mu(\xi_i^0-\eta_i^0g_1^{(\alpha)}-\omega_i^0g_1^{(\beta)}-\zeta_i^0g_1^{(\alpha)}-\psi_i^0g_1^{(\beta)})]u_i^0+\mu(\eta_i^0+\omega_i^0+\zeta_i^0g_2^{(\alpha)}+\psi_i^0g_2^{(\beta)})u_{i+1}^0+$$
$$\mu\eta_i^0\sum_{j=3}^{i+1}g_j^{(\alpha)}u_{i-j+1}^0+\mu\omega_i^0\sum_{j=3}^{i+1}g_j^{(\beta)}u_{i-j+1}^0+\mu\zeta_i^0\sum_{j=3}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^0+\mu\psi_i^0\sum_{j=3}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^0+\mu f_i^0+(1-\mu)f_i^1。\tag{14}$$

2）当 $n>0$ 时，有：

$$-(1-\mu)(\xi_i^n+\eta_i^n+\omega_i^n+\zeta_i^n+\psi_i^n)u_{i-1}^{n+1}+[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-\psi_i^ng_1^{(\beta)})]u_i^{n+1}-$$
$$(1-\mu)(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)})u_{i+1}^{n+1}-(1-\mu)\eta_i^n\sum_{j=3}^{i+1}g_j^{(\alpha)}u_{i-j+1}^{n+1}-(1-\mu)\omega_i^n\sum_{j=3}^{i+1}g_j^{(\beta)}u_{i-j+1}^{n+1}-$$
$$(1-\mu)\zeta_i^n\sum_{j=3}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^{n+1}-(1-\mu)\psi_i^n\sum_{j=3}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^{n+1}=\mu(\xi_i^n+\eta_i^ng_2^{(\alpha)}+\omega_i^ng_2^{(\beta)}+\zeta_i^n+\psi_i^n)u_{i-1}^n+$$
$$[\delta_1+\delta_2+\cdots+\delta_k-\delta_1(1-2^{r_1})-\delta_2(1-2^{r_2})-\cdots-\delta_k(1-2^{r_k})-\mu(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-\psi_i^ng_1^{(\beta)})]u_i^n+$$
$$\mu(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)})u_{i+1}^n+\mu\eta_i^n\sum_{j=3}^{i+1}g_j^{(\alpha)}u_{i-j+1}^n+\mu\omega_i^n\sum_{j=3}^{i+1}g_j^{(\beta)}u_{i-j+1}^n+\mu\zeta_i^n\sum_{j=3}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^n+$$
$$\mu\psi_i^n\sum_{j=3}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^n+\delta_1\sum_{j=1}^{n-1}d_j^{r_1}u_i^{n-j}+\delta_2\sum_{j=1}^{n-1}d_j^{r_2}u_i^{n-j}+\cdots+\delta_k\sum_{j=1}^{n-1}d_j^{r_k}u_i^{n-j}+\{\delta_1[(n+1)^{1-\gamma_1}-n^{1-\gamma_1}]+$$
$$\delta_2[(n+1)^{1-\gamma_2}-n^{1-\gamma_2}]+\cdots+\delta_k[(n+1)^{1-\gamma_k}-n^{1-\gamma_k}]\}u_i^0+\mu f_i^n+(1-\mu)f_i^{n+1}。\tag{15}$$

此时方程(1)的初边值条件为：$u_i^0=g(x_i)$，$i=0,1,2,\cdots,M$；$u_0^n=0$，$u_M^n=\varphi(t_n)$，$n=1,2,\cdots,N-1$。式中：

$$\delta_1=\theta_1\frac{\tau^{-\gamma_1}}{\Gamma(2-\gamma_1)},\delta_2=\theta_2\frac{\tau^{-\gamma_2}}{\Gamma(2-\gamma_2)},\cdots,\delta_k=\theta_k\frac{\tau^{-\gamma_k}}{\Gamma(2-\gamma_k)},\xi_i^n=\frac{v_i^n}{h},\eta_i^n=\frac{a_{+i}^n}{h^\alpha},\zeta_i^n=\frac{a_{-i}^n}{h^\alpha},\omega_i^n=\frac{b_{+i}^n}{h^\beta},$$
$$\psi_i^n=\frac{b_{-i}^n}{h^\beta},d_j^{\gamma_1}=2(j+1)^{1-\gamma_1}-(j+2)^{1-\gamma_1}-j^{1-\gamma_1},d_j^{\gamma_2}=2(j+1)^{1-\gamma_2}-(j+2)^{1-\gamma_2}-j^{1-\gamma_2},\cdots,$$
$$d_j^{\gamma_k}=2(j+1)^{1-\gamma_k}-(j+2)^{1-\gamma_k}-j^{1-\gamma_k},j=1,2,\cdots,n-1。$$

方程(14)和(15)可以写成如下矩阵形式：

$$\begin{cases}\boldsymbol{A}_1\boldsymbol{U}^1=\boldsymbol{B}_0\boldsymbol{U}^0+\boldsymbol{Q}^0,\\ \boldsymbol{A}_{n+1}\boldsymbol{U}^{n+1}=\boldsymbol{B}_n\boldsymbol{U}^n+(\delta_1d_1^{\gamma_1}+\delta_2d_1^{\gamma_2}+\cdots+\delta_kd_1^{\gamma_k})\boldsymbol{U}^{n-1}+\cdots+(\delta_1d_{n-1}^{\gamma_1}+\delta_2d_{n-1}^{\gamma_2}+\cdots+\delta_kd_{n-1}^{\gamma_k})\boldsymbol{U}^1+\\ \quad\{\delta_1[(n+1)^{1-\gamma_1}-n^{1-\gamma_1}]+\delta_2[(n+1)^{1-\gamma_2}-n^{1-\gamma_2}]+\cdots+\delta_k[(n+1)^{1-\gamma_k}-n^{1-\gamma_k}]\}\boldsymbol{U}^0+\boldsymbol{Q}^n。\end{cases}$$

式中：

$$\boldsymbol{U}^n=(u_1^n,u_2^n,\cdots,u_{M-1}^n)^{\mathrm{T}},\boldsymbol{U}^0=[g(x_1),g(x_2),\cdots,g(x_{M-1})]^{\mathrm{T}},$$
$$b=(\eta_{M-1}^n+\omega_{M-1}^n+\zeta_{M-1}^ng_2^{(\alpha)}+\psi_{M-1}^ng_2^{(\beta)})\times[(1-\theta)u_M^{n+1}+\theta u_M^n],$$
$$\boldsymbol{A}_n=(a_{i,j}^n)_{(M-1)\times(M-1)},\boldsymbol{Q}^n=\mu\boldsymbol{F}^n+(1-\mu)\boldsymbol{F}^{n+1}+(1-\mu)\boldsymbol{U}_M^{n+1}\boldsymbol{E}+\mu\boldsymbol{U}_M^n\boldsymbol{E},$$
$$\boldsymbol{F}^n=(f_1^n,f_2^n,\cdots,f_{M-1}^n+b)^{\mathrm{T}},\boldsymbol{E}=(\zeta_1^ng_M^{(\alpha)}+\psi_1^ng_M^{(\beta)},\zeta_2^ng_{M-1}^{(\alpha)}+\psi_2^ng_{M-1}^{(\beta)},\cdots,\zeta_{M-1}^ng_2^{(\alpha)}+\psi_{M-1}^ng_2^{(\beta)})^{\mathrm{T}}。$$

且：

$$a_{i,j}^n=\begin{cases}-(1-\mu)(\xi_i^n+\eta_i^n+\omega_i^n+\zeta_i^n+\psi_i^n),j=i-1,\\ [\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-\psi_i^ng_1^{(\beta)})],j=i,\\ -(1-\mu)(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)}),j=i+1,\\ -(1-\mu)(\eta_i^ng_{i+1-j}^{(\alpha)}+\omega_i^ng_{i+1-j}^{(\beta)}),j=1,2,\cdots,i-2,\\ -(1-\mu)(\zeta_i^ng_{j+1-i}^{(\alpha)}+\psi_i^ng_{j+1-i}^{(\beta)}),j=i+2,i+3,\cdots,M-1。\end{cases}$$

可以验证矩阵 $\boldsymbol{A}_n$ 是严格对角占优的，所以方程(14)和(15)有唯一解。

2　差分格式的稳定性

引理1[27]　对于任意的正整数 m，有 $\sum_{j=0}^{m}g_j<0$。

定理 1　当 $\mu\left\{\frac{1}{h}\max\limits_{x\in[0,L],t\in[0,T]}v(x,t)+\frac{\alpha}{h^{\alpha}}\max\limits_{x\in[0,L],t\in[0,T]}[a_{+}(x,t)+a_{-}(x,t)]+\frac{\beta}{h^{\beta}}\max\limits_{x\in[0,L],t\in[0,T]}[(b_{+}(x,t)+b_{-}(x,t)]\right\}\leqslant\delta_1+\delta_2+\cdots+\delta_k-\delta_1(1-2^{r_1})-\delta_2(1-2^{r_2})-\cdots-\delta_k(1-2^{r_k})$时,加权隐式差分格式(10)关于初值条件稳定。

证明　定义两差分算子 L_1,L_2 分别为:

$$L_1u_i^{n+1}=-(1-\mu)(\xi_i^n+\eta_i^n+\omega_i^n+\zeta_i^n+\psi_i^n)u_{i-1}^{n+1}+[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-\psi_i^ng_1^{(\beta)})]u_i^{n+1}-(1-\mu)(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)})u_{i+1}^{n+1}-(1-\mu)\eta_i^n\sum_{j=3}^{i+1}g_j^{(\alpha)}u_{i-j+1}^{n+1}-(1-\mu)\omega_i^n\sum_{j=3}^{i+1}g_j^{(\beta)}u_{i-j+1}^{n+1}-(1-\mu)\zeta_i^n\sum_{j=3}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^{n+1}-(1-\mu)\psi_i^n\sum_{j=3}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^{n+1}。\tag{16}$$

$$L_2u_i^n=\mu(\xi_i^n+\eta_i^ng_2^{(\alpha)}+\omega_i^ng_2^{(\beta)}+\zeta_i^n+\psi_i^n)u_{i-1}^n+[\delta_1+\delta_2+\cdots+\delta_k-\delta_1(1-2^{r_1})-\delta_2(1-2^{r_2})-\cdots-\delta_k(1-2^{r_k})-\mu(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-\psi_i^ng_1^{(\beta)})]u_i^n+\mu(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)})u_{i+1}^n+\mu\eta_i^n\sum_{j=3}^{i+1}g_j^{(\alpha)}u_{i-j+1}^n+\mu\omega_i^n\sum_{j=3}^{i+1}g_j^{(\beta)}u_{i-j+1}^n+\mu\zeta_i^n\sum_{j=3}^{M-i+1}g_j^{(\alpha)}u_{i+j-1}^n+\mu\psi_i^n\sum_{j=3}^{M-i+1}g_j^{(\beta)}u_{i+j-1}^n+\delta_1\sum_{j=1}^{n-1}d_j^{r_1}u_i^{n-j}+\delta_2\sum_{j=1}^{n-1}d_j^{r_2}u_i^{n-j}+\cdots+\delta_k\sum_{j=1}^{n-1}d_j^{r_k}u_i^{n-j}+\{\delta_1[(n+1)^{1-\gamma_1}-n^{1-\gamma_1}]+\delta_2[(n+1)^{1-\gamma_2}-n^{1-\gamma_2}]+\cdots+\delta_k[(n+1)^{1-\gamma_k}-n^{1-\gamma_k}]\}u_i^0。\tag{17}$$

则加权隐式差分格式(10)可以写成:

$$L_1u_i^{n+1}=L_2u_i^n+\mu f_i^n+(1-\mu)f_i^{n+1}。\tag{18}$$

假设 $\tilde{u}_i^n,u_i^n(i=1,2,\cdots,M-1;n=1,2,\cdots,N-1)$分别是关于初值 $g_1(x),g_2(x)$的满足方程(10)的解,假定 f_i^n 的计算是精确的,则计算误差 $\varepsilon_i^n=\tilde{u}_i^n-u_i^n$ 满足:

$$L_1\varepsilon_i^{n+1}=L_2\varepsilon_i^n。\tag{19}$$

即:

1) 当 $n=0$ 时,有:

$$-(1-\mu)(\xi_i^0+\eta_i^0g_2^{(\alpha)}+\omega_i^0g_2^{(\beta)}+\zeta_i^0+\psi_i^0)\varepsilon_{i-1}^1+[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_i^0-\eta_i^0g_1^{(\alpha)}-\omega_i^0g_1^{(\beta)}-\zeta_i^0g_1^{(\alpha)}-\psi_i^0g_1^{(\beta)})]\varepsilon_i^1-(1-\mu)(\eta_i^0+\omega_i^0+\zeta_i^0g_2^{(\alpha)}+\psi_i^0g_2^{(\beta)})\varepsilon_{i+1}^1-(1-\mu)\eta_i^0\sum_{j=3}^{i+1}g_j^{(\alpha)}\varepsilon_{i-j+1}^1-(1-\mu)\omega_i^0\sum_{j=3}^{i+1}g_j^{(\beta)}\varepsilon_{i-j+1}^1-(1-\mu)\zeta_i^0\sum_{j=3}^{M-i+1}g_j^{(\alpha)}\varepsilon_{i+j-1}^1-(1-\mu)\psi_i^0\sum_{j=3}^{M-i+1}g_j^{(\beta)}\varepsilon_{i+j-1}^1=\mu(\xi_i^0+\eta_i^0g_2^{(\alpha)}+\omega_i^0g_2^{(\beta)}+\zeta_i^0+\psi_i^0)\varepsilon_{i-1}^0+[\delta_1+\delta_2+\cdots+\delta_k-\mu(\xi_i^0-\eta_i^0g_1^{(\alpha)}-\omega_i^0g_1^{(\beta)}-\zeta_i^0g_1^{(\alpha)}-\psi_i^0g_1^{(\beta)})]\varepsilon_i^0+\mu(\eta_i^0+\omega_i^0+\zeta_i^0g_2^{(\alpha)}+\psi_i^0g_2^{(\beta)})\varepsilon_{i+1}^0+\mu\eta_i^0\sum_{j=3}^{i+1}g_j^{(\alpha)}\varepsilon_{i-j+1}^0+\mu\omega_i^0\sum_{j=3}^{i+1}g_j^{(\beta)}\varepsilon_{i-j+1}^0+\mu\zeta_i^0\sum_{j=3}^{M-i+1}g_j^{(\alpha)}\varepsilon_{i+j-1}^0+\mu\psi_i^0\sum_{j=3}^{M-i+1}g_j^{(\beta)}\varepsilon_{i+j-1}^0。\tag{20}$$

2) 当 $n>0$ 时,有:

$$-(1-\mu)(\xi_i^n+\eta_i^n+\omega_i^n+\zeta_i^n+\psi_i^n)\varepsilon_{i-1}^{n+1}+[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-\psi_i^ng_1^{(\beta)})]\varepsilon_i^{n+1}-(1-\mu)(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)})\varepsilon_{i+1}^{n+1}-(1-\mu)\eta_i^n\sum_{j=3}^{i+1}g_j^{(\alpha)}\varepsilon_{i-j+1}^{n+1}-(1-\mu)\omega_i^n\sum_{j=3}^{i+1}g_j^{(\beta)}\varepsilon_{i-j+1}^{n+1}-(1-\mu)\zeta_i^n\sum_{j=3}^{M-i+1}g_j^{(\alpha)}\varepsilon_{i+j-1}^{n+1}-(1-\mu)\psi_i^n\sum_{j=3}^{M-i+1}g_j^{(\beta)}\varepsilon_{i+j-1}^{n+1}=\mu(\xi_i^n+\eta_i^ng_2^{(\alpha)}+\omega_i^ng_2^{(\beta)}+\zeta_i^n+\psi_i^n)\varepsilon_{i-1}^n+[\delta_1+\delta_2+\cdots+\delta_k-\delta_1(1-2^{r_1})-\delta_2(1-2^{r_2})-\cdots-\delta_k(1-2^{r_k})-\mu(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-\psi_i^ng_1^{(\beta)})]\varepsilon_i^n+\mu(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)})\varepsilon_{i+1}^n+\mu\eta_i^n\sum_{j=3}^{i+1}g_j^{(\alpha)}\varepsilon_{i-j+1}^n+\mu\omega_i^n\sum_{j=3}^{i+1}g_j^{(\beta)}\varepsilon_{i-j+1}^n+\mu\zeta_i^n\sum_{j=3}^{M-i+1}g_j^{(\alpha)}\varepsilon_{i+j-1}^n+\mu\psi_i^n\sum_{j=3}^{M-i+1}g_j^{(\beta)}\varepsilon_{i+j-1}^n+\delta_1\sum_{j=1}^{n-1}d_j^{r_1}\varepsilon_i^{n-j}+\delta_2\sum_{j=1}^{n-1}d_j^{r_2}\varepsilon_i^{n-j}+\cdots+\delta_k\sum_{j=1}^{n-1}d_j^{r_k}\varepsilon_i^{n-j}+\{\delta_1[(n+1)^{1-\gamma_1}-n^{1-\gamma_1}]+$$

$$\delta_2[(n+1)^{1-\gamma_2}-n^{1-\gamma_2}]+\cdots+\delta_k[(n+1)^{1-\gamma_k}-n^{1-\gamma_k}]\}\varepsilon_i^0。\quad(21)$$

方程(20)和(21)也可写成：

$$\begin{cases}\boldsymbol{A}_1\boldsymbol{E}^1=\boldsymbol{B}_0\boldsymbol{E}^0,\\ \boldsymbol{A}_{n+1}\boldsymbol{E}^{n+1}=\boldsymbol{B}_n\boldsymbol{E}^n+(\delta_1d_1^{\gamma_1}+\delta_2d_1^{\gamma_2}+\cdots+\delta_kd_1^{\gamma_k})\boldsymbol{E}^{n-1}+\cdots+(\delta_1d_{n-1}^{\gamma_1}+\delta_2d_{n-1}^{\gamma_2}+\cdots+\delta_kd_{n-1}^{\gamma_k})\boldsymbol{E}^1+\\ \quad\{\delta_1[(n+1)^{1-\gamma_1}-n^{1-\gamma_1}]+\delta_2[(n+1)^{1-\gamma_2}-n^{1-\gamma_2}]+\cdots+\delta_k[(n+1)^{1-\gamma_k}-n^{1-\gamma_k}]\}\boldsymbol{E}^0。\end{cases}$$

式中:误差向量 $\boldsymbol{E}^n=(\varepsilon_1^n,\varepsilon_2^n,\cdots,\varepsilon_{M-1}^n)^{\mathrm{T}}$。

下用数学归纳法证明 $\|\boldsymbol{E}^n\|_\infty\leqslant\|\boldsymbol{E}^0\|_\infty,n=0,1,2,\cdots,N-1$。

1) 当 $n=1$ 时,记 $|\varepsilon_l^1|=\max\limits_{1\leqslant i\leqslant M-1}|\varepsilon_i^1|$。因为 $\delta_1>0,\delta_2>0,\cdots,\delta_k>0,\xi_i^n>0,\eta_i^n>0,\omega_i^n>0,\zeta_i^n>0,\psi_i^n>0$,且由引理1,对任意的正整数 m,有 $\sum\limits_{j=0}^{m}g_j\leqslant0$,因此:

$$\begin{aligned}&\|\boldsymbol{E}^1\|_\infty=|\varepsilon_l^1|\leqslant-(1-\mu)(\xi_l^0+\eta_l^0g_2^{(\alpha)}+\omega_l^0g_2^{(\beta)}+\zeta_l^0+\psi_l^0)|\varepsilon_{l-1}^1|+\\&[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_l^0-\eta_l^0g_1^{(\alpha)}-\omega_l^0g_1^{(\beta)}-\zeta_l^0g_1^{(\alpha)}-\psi_l^0g_1^{(\beta)})]|\varepsilon_l^1|-\\&(1-\mu)(\eta_l^0+\omega_l^0+\zeta_l^0g_2^{(\alpha)}+\psi_l^0g_2^{(\beta)})|\varepsilon_{l+1}^1|-(1-\mu)\eta_l^0\sum_{j=3}^{l+1}g_j^{(\alpha)}|\varepsilon_{l-j+1}^1|-(1-\mu)\omega_l^0\sum_{j=3}^{l+1}g_j^{(\beta)}|\varepsilon_{l-j+1}^1|-\\&(1-\mu)\zeta_l^0\sum_{j=3}^{M-l+1}g_j^{(\alpha)}|\varepsilon_{l+j-1}^1|-(1-\mu)\psi_l^0\sum_{j=3}^{M-l+1}g_j^{(\beta)}|\varepsilon_{l+j-1}^1|=\mu(\xi_l^0+\eta_l^0g_2^{(\alpha)}+\omega_l^0g_2^{(\beta)}+\zeta_l^0+\psi_l^0)|\varepsilon_{l-1}^0|+\\&[\delta_1+\delta_2+\cdots+\delta_k-\mu(\xi_l^0-\eta_l^0g_1^{(\alpha)}-\omega_l^0g_1^{(\beta)}-\zeta_l^0g_1^{(\alpha)}-\psi_l^0g_1^{(\beta)})]|\varepsilon_l^0|+\mu(\eta_l^0+\omega_l^0+\zeta_l^0g_2^{(\alpha)}+\psi_l^0g_2^{(\beta)})|\varepsilon_{l+1}^0|+\\&\mu\eta_l^0\sum_{j=3}^{l+1}g_j^{(\alpha)}|\varepsilon_{l-j+1}^0|+\mu\omega_l^0\sum_{j=3}^{l+1}g_j^{(\beta)}|\varepsilon_{l-j+1}^0|+\mu\zeta_l^0\sum_{j=3}^{M-l+1}g_j^{(\alpha)}|\varepsilon_{l+j-1}^0|+\mu\psi_l^0\sum_{j=3}^{M-l+1}g_j^{(\beta)}|\varepsilon_{l+j-1}^0|\leqslant\\&|\mu(\xi_l^0+\eta_l^0g_2^{(\alpha)}+\omega_l^0g_2^{(\beta)}+\zeta_l^0+\psi_l^0)\varepsilon_{l-1}^0+[\delta_1+\delta_2+\cdots+\delta_k-\mu(\xi_l^0-\eta_l^0g_1^{(\alpha)}-\omega_l^0g_1^{(\beta)}-\zeta_l^0g_1^{(\alpha)}-\psi_l^0g_1^{(\beta)})]\varepsilon_l^0+\\&\mu(\eta_l^0+\omega_l^0+\zeta_l^0g_2^{(\alpha)}+\psi_l^0g_2^{(\beta)})\varepsilon_{l+1}^0+\mu\eta_l^0\sum_{j=3}^{l+1}g_j^{(\alpha)}\varepsilon_{l-j+1}^0+\mu\omega_l^0\sum_{j=3}^{l+1}g_j^{(\beta)}\varepsilon_{l-j+1}^0+\\&\mu\zeta_l^0\sum_{j=3}^{M-l+1}g_j^{(\alpha)}\varepsilon_{l+j-1}^0+\mu\psi_l^0\sum_{j=3}^{M-l+1}g_j^{(\beta)}\varepsilon_{l+j-1}^0|=|L_1\varepsilon_l^1|=|L_2\varepsilon_l^0|=|\varepsilon_l^0|\leqslant\|\boldsymbol{E}^0\|_\infty。\end{aligned}$$

因此,$\|\boldsymbol{E}^1\|_\infty\leqslant\|\boldsymbol{E}^0\|_\infty$。

2) 假设当 $n\leqslant s$ 时都有 $\|\boldsymbol{E}^n\|_\infty\leqslant\|\boldsymbol{E}^0\|_\infty,n=1,2,\cdots,s$,记 $|\varepsilon_l^{s+1}|=\max\limits_{1\leqslant i\leqslant M-1}|\varepsilon_i^{s+1}|$,则当 $n=s+1$ 时,有:

$$\begin{aligned}&\|\boldsymbol{E}^{s+1}\|_\infty=|\varepsilon_l^{s+1}|\leqslant-(1-\mu)(\xi_l^{s+1}+\eta_l^{s+1}g_2^{(\alpha)}+\omega_l^{s+1}g_2^{(\beta)}+\zeta_l^{s+1}+\psi_l^{s+1})|\varepsilon_{l-1}^{s+1}|+\\&[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_l^{s+1}-\eta_l^{s+1}g_1^{(\alpha)}-\omega_l^{s+1}g_1^{(\beta)}-\zeta_l^{s+1}g_1^{(\alpha)}-\psi_l^{s+1}g_1^{(\beta)})]|\varepsilon_l^{s+1}|-\\&(1-\mu)(\eta_l^{s+1}+\omega_l^{s+1}+\zeta_l^{s+1}g_2^{(\alpha)}+\psi_l^{s+1}g_2^{(\beta)})|\varepsilon_{l+1}^{s+1}|-(1-\mu)\eta_l^{s+1}\sum_{j=3}^{l+1}g_j^{(\alpha)}|\varepsilon_{l-j+1}^{s+1}|-\\&(1-\mu)\omega_l^{s+1}\sum_{j=3}^{l+1}g_j^{(\beta)}|\varepsilon_{l-j+1}^{s+1}|-(1-\mu)\zeta_l^{s+1}\sum_{j=3}^{M-l+1}g_j^{(\alpha)}|\varepsilon_{l+j-1}^{s+1}|-(1-\mu)\psi_l^{s+1}\sum_{j=3}^{M-l+1}g_j^{(\beta)}|\varepsilon_{l+j-1}^{s+1}|=\\&\mu(\xi_l^{s+1}+\eta_l^{s+1}g_2^{(\alpha)}+\omega_l^{s+1}g_2^{(\beta)}+\zeta_l^{s+1}+\psi_l^{s+1})|\varepsilon_{l-1}^{s+1}|+[\delta_1+\delta_2+\cdots+\delta_k-\delta_1(1-2^{r_1})-\delta_2(1-2^{r_2})-\cdots-\\&\delta_k(1-2^{r_k})-\mu(\xi_l^{s+1}-\eta_l^{s+1}g_1^{(\alpha)}-\omega_l^{s+1}g_1^{(\beta)}-\zeta_l^{s+1}g_1^{(\alpha)}-\psi_l^{s+1}g_1^{(\beta)})]|\varepsilon_l^{s+1}|+\mu(\eta_l^{s+1}+\omega_l^{s+1}+\zeta_l^{s+1}g_2^{(\alpha)}+\\&\psi_l^{s+1}g_2^{(\beta)})|\varepsilon_{l+1}^{s+1}|+\mu\eta_l^{s+1}\sum_{j=3}^{l+1}g_j^{(\alpha)}|\varepsilon_{l-j+1}^{s+1}|+\mu\omega_l^{s+1}\sum_{j=3}^{l+1}g_j^{(\beta)}|\varepsilon_{l-j+1}^{s+1}|+\mu\zeta_l^{s+1}\sum_{j=3}^{M-l+1}g_j^{(\alpha)}|\varepsilon_{l+j-1}^{s+1}|+\\&\mu\psi_l^{s+1}\sum_{j=3}^{M-l+1}g_j^{(\beta)}|\varepsilon_{l+j-1}^{s+1}|+\delta_1\sum_{j=1}^{s-1}d_j^{r_1}|\varepsilon_l^{s+1-j}|+\delta_2\sum_{j=1}^{s-1}d_j^{r_2}|\varepsilon_l^{s+1-j}|+\cdots+\delta_k\sum_{j=1}^{s-1}d_j^{r_k}|\varepsilon_l^{s+1-j}|+\\&\{\delta_1[(s+2)^{1-\gamma_1}-(s+1)^{1-\gamma_1}]+\delta_2[(s+2)^{1-\gamma_2}-(s+1)^{1-\gamma_2}]+\cdots+\delta_k[(s+2)^{1-\gamma_k}-(s+1)^{1-\gamma_k}]\}|\varepsilon_l^0|\leqslant\\&|\mu(\xi_l^{s+1}+\eta_l^{s+1}g_2^{(\alpha)}+\omega_l^{s+1}g_2^{(\beta)}+\zeta_l^{s+1}+\psi_l^{s+1})\varepsilon_{l-1}^{s+1}+[\delta_1+\delta_2+\cdots+\delta_k-\delta_1(1-2^{r_1})-\delta_2(1-2^{r_2})-\cdots-\\&\delta_k(1-2^{r_k})-\mu(\xi_l^{s+1}-\eta_l^{s+1}g_1^{(\alpha)}-\omega_l^{s+1}g_1^{(\beta)}-\zeta_l^{s+1}g_1^{(\alpha)}-\psi_l^{s+1}g_1^{(\beta)})]\varepsilon_l^{s+1}+\mu(\eta_l^{s+1}+\omega_l^{s+1}+\zeta_l^{s+1}g_2^{(\alpha)}+\\&\psi_l^{s+1}g_2^{(\beta)})\varepsilon_{l+1}^{s+1}+\mu\eta_l^{s+1}\sum_{j=3}^{l+1}g_j^{(\alpha)}\varepsilon_{l-j+1}^{s+1}+\mu\omega_l^{s+1}\sum_{j=3}^{l+1}g_j^{(\beta)}\varepsilon_{l-j+1}^{s+1}+\mu\zeta_l^{s+1}\sum_{j=3}^{M-l+1}g_j^{(\alpha)}\varepsilon_{l+j-1}^{s+1}+\mu\psi_l^{s+1}\sum_{j=3}^{M-l+1}g_j^{(\beta)}\varepsilon_{l+j-1}^{s+1}+\end{aligned}$$

$$\delta_1\sum_{j=1}^{s-1}d_j^{r_1}\varepsilon_l^{s+1-j}+\delta_2\sum_{j=1}^{s-1}d_j^{r_2}\varepsilon_l^{s+1-j}+\cdots+\delta_k\sum_{j=1}^{s-1}d_j^{r_k}\varepsilon_l^{s+1-j}+\{\delta_1[(s+2)^{1-\gamma_1}-(s+1)^{1-\gamma_1}]+$$
$$\delta_2[(s+2)^{1-\gamma_2}-(s+1)^{1-\gamma_2}]+\cdots+\delta_k[(s+2)^{1-\gamma_k}-(s+1)^{1-\gamma_k}]\}\varepsilon_l^0|=|L_1\varepsilon_l^{s+1}|=|L_2\varepsilon_l^s|=|\varepsilon_l^s|\leqslant$$
$$\{\mu(\xi_l^{s+1}+\eta_l^{s+1}g_2^{(\alpha)}+\omega_l^{s+1}g_2^{(\beta)}+\zeta_l^{s+1}+\psi_l^{s+1})+[\delta_1+\delta_2+\cdots+\delta_k-\delta_1(1-2^{r_1})-\delta_2(1-2^{r_2})-\cdots-$$
$$\delta_k(1-2^{r_k})-\mu(\xi_l^{s+1}-\eta_l^{s+1}g_1^{(\alpha)}-\omega_l^{s+1}g_1^{(\beta)}-\zeta_l^{s+1}g_1^{(\alpha)}-\psi_l^{s+1}g_1^{(\beta)})]+\mu(\eta_l^{s+1}+\omega_l^{s+1}+\zeta_l^{s+1}g_2^{(\alpha)}+\psi_l^{s+1}g_2^{(\beta)})+$$
$$\mu\eta_l^{s+1}\sum_{j=3}^{l+1}g_j^{(\alpha)}+\mu\omega_l^{s+1}\sum_{j=3}^{l+1}g_j^{(\beta)}+\mu\zeta_l^{s+1}\sum_{j=3}^{M-l+1}g_j^{(\alpha)}+\mu\psi_l^{s+1}\sum_{j=3}^{M-l+1}g_j^{(\beta)}+\delta_1\sum_{j=1}^{s-1}d_j^{r_1}+\delta_2\sum_{j=1}^{s-1}d_j^{r_2}+\cdots+\delta_k\sum_{j=1}^{s-1}d_j^{r_k}+$$
$$\{\delta_1[(s+2)^{1-\gamma_1}-(s+1)^{1-\gamma_1}]+\delta_2[(s+2)^{1-\gamma_2}-(s+1)^{1-\gamma_2}]+\cdots+$$
$$\delta_k[(s+2)^{1-\gamma_k}-(s+1)^{1-\gamma_k}]\}\}\|\boldsymbol{E}^0\|_\infty\leqslant\|\boldsymbol{E}^0\|_\infty。$$

所以当 $n=s+1$ 时，$\|\boldsymbol{E}^{s+1}\|_\infty\leqslant\|\boldsymbol{E}^0\|_\infty$ 也成立。因此加权隐式差分格式(10)关于初值条件稳定。　证毕

3　差分格式的收敛性

引理 2[29]　当 $0<\gamma<1$ 时，系数 $\sigma_l=(l+1)^{1-\gamma}-l^{1-\gamma}$ 满足：

$$\sigma_l>0,l=0,1,\cdots;\sigma_l>\sigma_{l+1},l=0,1,\cdots;\sigma_{l-1}>\frac{1-\gamma}{l^\gamma},l=1,2,\cdots$$

引理 3　设 $h(x)=(x+1)^{1-\gamma}-x^{1-\gamma}(0<\gamma<1,x\geqslant0)$，则 $h(x)$ 单调递减且 $h(x+1)-2h(x)+h(x-1)>0\ (x>1)$。

设 $u(x_i,t_n)(i=1,2,\cdots,M-1;n=1,2,\cdots,N-1)$ 是微分方程在网格点(x_i,t_n)上的精确解。定义该精确解与加权隐式差分格式(10)的数值解的误差为 $e_i^n=u(x_i,t_n)-u_i^n$ 及 $\boldsymbol{e}^n=(e_1^n,e_2^n,\cdots,e_{M-1}^n)^{\mathrm{T}}$，显然 $\boldsymbol{e}^0=\boldsymbol{0}$，且误差满足方程：

$$\begin{cases}L_1e_i^{n+1}=L_2e_i^n+R_i^{n+1},\\ e_i^0=0。\end{cases}$$

式中：$i=1,2,\cdots,M-1;n=0,1,2,\cdots,N-1;R_i^{n+1}$ 是局部截断误差且 $|R_i^{n+1}|\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)$；$C$ 是常数。将 $u_i^n=u(x_i,t_n)-e_i^n$ 代入差分格式(14)和(15)，可得：

1）当 $n=0$ 时，有：

$$R_i^1=-(1-\mu)(\xi_i^0+\eta_i^0g_2^{(\alpha)}+\omega_i^0g_2^{(\beta)}+\zeta_i^0+\psi_i^0)e_{i-1}^1+$$
$$[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_i^0-\eta_i^0g_1^{(\alpha)}-\omega_i^0g_1^{(\beta)}-\zeta_i^0g_1^{(\alpha)}-\psi_i^0g_1^{(\beta)})]e_i^1-$$
$$(1-\mu)(\eta_i^0+\omega_i^0+\zeta_i^0g_2^{(\alpha)}+\psi_i^0g_2^{(\beta)})e_{i+1}^1-(1-\mu)\eta_i^0\sum_{j=3}^{i+1}g_j^{(\alpha)}e_{i-j+1}^1-(1-\mu)\omega_i^0\sum_{j=3}^{i+1}g_j^{(\beta)}e_{i-j+1}^1-$$
$$(1-\mu)\zeta_i^0\sum_{j=3}^{M-i+1}g_j^{(\alpha)}e_{i+j-1}^1-(1-\mu)\psi_i^0\sum_{j=3}^{M-i+1}g_j^{(\beta)}e_{i+j-1}^1。$$

2）当 $n>0$ 时，有：

$$R_i^{n+1}=-(1-\mu)(\xi_i^n+\eta_i^n+\omega_i^n+\zeta_i^n+\psi_i^n)e_{i-1}^{n+1}+[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-$$
$$\psi_i^ng_1^{(\beta)})]e_i^{n+1}-(1-\mu)(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)})e_{i+1}^{n+1}-(1-\mu)\eta_i^n\sum_{j=3}^{i+1}g_j^{(\alpha)}e_{i-j+1}^{n+1}-(1-\mu)\omega_i^n\sum_{j=3}^{i+1}g_j^{(\beta)}e_{i-j+1}^{n+1}-$$
$$(1-\mu)\zeta_i^n\sum_{j=3}^{M-i+1}g_j^{(\alpha)}e_{i+j-1}^{n+1}-(1-\mu)\psi_i^n\sum_{j=3}^{M-i+1}g_j^{(\beta)}e_{i+j-1}^{n+1}-\mu(\xi_i^n+\eta_i^ng_2^{(\alpha)}+\omega_i^ng_2^{(\beta)}+\zeta_i^n+\psi_i^n)e_{i-1}^n-$$
$$[\delta_1+\delta_2+\cdots+\delta_k-\delta_1(1-2^{r_1})-\delta_2(1-2^{r_2})-\cdots-\delta_k(1-2^{r_k})-\mu(\xi_i^n-\eta_i^ng_1^{(\alpha)}-\omega_i^ng_1^{(\beta)}-\zeta_i^ng_1^{(\alpha)}-\psi_i^ng_1^{(\beta)})]e_i^n-$$
$$\mu(\eta_i^n+\omega_i^n+\zeta_i^ng_2^{(\alpha)}+\psi_i^ng_2^{(\beta)})e_{i+1}^n-\mu\eta_i^n\sum_{j=3}^{i+1}g_j^{(\alpha)}e_{i-j+1}^n-\mu\omega_i^n\sum_{j=3}^{i+1}g_j^{(\beta)}e_{i-j+1}^n-$$
$$\mu\zeta_i^n\sum_{j=3}^{M-i+1}g_j^{(\alpha)}e_{i+j-1}^n-\mu\psi_i^n\sum_{j=3}^{M-i+1}g_j^{(\beta)}e_{i+j-1}^n-\delta_1\sum_{j=1}^{n-1}d_j^{r_1}e_i^{n-j}-\delta_2\sum_{j=1}^{n-1}d_j^{r_2}e_i^{n-j}-\cdots-\delta_k\sum_{j=1}^{n-1}d_j^{r_k}e_i^{n-j}。$$

定理 2　加权隐式差分格式(10)的数值解与精确解的误差满足 $\|\boldsymbol{e}^n\|_\infty\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\sigma_{n-1}^{-1}$，其中 $\sigma_{n-1}=\delta_1[(n+1)^{1-\gamma_1}-n^{1-\gamma_1}]+\delta_2[(n+1)^{1-\gamma_2}-n^{1-\gamma_2}]+\cdots+\delta_k[(n+1)^{1-\gamma_k}-n^{1-\gamma_k}],n=1,2,\cdots,N$。

证明 用数学归纳法证明。当 $n=0$ 时,设 $|e_l^1|=\max\limits_{1\leqslant i\leqslant M-1}|e_i^1|$,得到:

$$\|\boldsymbol{e}^1\|_\infty=|e_l^1|\leqslant -(1-\mu)(\xi_l^0+\eta_l^0 g_2^{(\alpha)}+\omega_l^0 g_2^{(\beta)}+\zeta_l^0+\psi_l^0)|e_{l-1}^1|+[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_l^0-\eta_l^0 g_1^{(\alpha)}-\omega_l^0 g_1^{(\beta)}-\zeta_l^0 g_1^{(\alpha)}-\psi_l^0 g_1^{(\beta)})]|e_l^1|-(1-\mu)(\eta_l^0+\omega_l^0+\zeta_l^0 g_2^{(\alpha)}+\psi_l^0 g_2^{(\beta)})|e_{l+1}^1|-(1-\mu)\eta_l^0\sum_{j=3}^{l+1}g_j^{(\alpha)}|e_{l-j+1}^1|-(1-\mu)\psi_l^0\sum_{j=3}^{l+1}g_j^{(\beta)}|e_{l-j+1}^1|-(1-\mu)\zeta_l^0\sum_{j=3}^{M-l+1}g_j^{(\alpha)}|e_{l+j-1}^1|-(1-\mu)\psi_l^0\sum_{j=3}^{M-l+1}g_j^{(\beta)}|e_{l+j-1}^1|\leqslant(\eta_l^0 g_2^{(\alpha)}+\omega_l^0 g_2^{(\beta)}+\zeta_l^0+\psi_l^0)e_{l-1}^1+|-(1-\mu)(\xi_l^0+[\delta_1+\delta_2+\cdots+\delta_k+(1-\mu)(\xi_l^0-\eta_l^0 g_1^{(\alpha)}-\omega_l^0 g_1^{(\beta)}-\zeta_l^0 g_1^{(\alpha)}-\psi_l^0 g_1^{(\beta)})]e_l^1-(1-\mu)(\eta_l^0+\omega_l^0+\zeta_l^0 g_2^{(\alpha)}+\psi_l^0 g_2^{(\beta)})e_{l+1}^1-(1-\mu)\eta_l^0\sum_{j=3}^{l+1}g_j^{(\alpha)}e_{l-j+1}^1-(1-\mu)\omega_l^0\sum_{j=3}^{l+1}g_j^{(\beta)}e_{l-j+1}^1-(1-\mu)\zeta_l^0\sum_{j=3}^{M-l+1}g_j^{(\alpha)}e_{l+j-1}^1|-(1-\mu)\psi_l^0\sum_{j=3}^{M-l+1}g_j^{(\beta)}e_{l+j-1}^1|=|L_1e_l^1|=|L_2e_l^0+R_l^1|=|e_l^0+R_l^1|=|R_l^1|\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\sigma_0^{-1}。$$

因此,$\|\boldsymbol{e}^1\|_\infty\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\sigma_0^{-1}$。

假设当 $n\leqslant s$ 时都有 $\|\boldsymbol{e}^n\|_\infty\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\sigma_{n-1}^{-1}$ 成立,$n=1,2,\cdots,s$,则当 $n=s+1$ 时,设 $|e_l^{s+1}|=\max\limits_{1\leqslant i\leqslant M-1}|e_i^{s+1}|$注意到$\sigma_j^{-1}\leqslant\sigma_n^{-1}$,$j=0,1,2,\cdots,n$ 及 $\sum\limits_{j=0}^{i}g_j<0$,$i=1,2,\cdots,M$。于是

$$\|\boldsymbol{e}^{s+1}\|_\infty=|e_l^{s+1}|\leqslant(\delta_1 d_1^{\gamma_1}+\delta_2 d_1^{\gamma_2}+\cdots+\delta_k d_1^{\gamma_k})\|\boldsymbol{e}^s\|_\infty+\left(\delta_1\sum_{j=1}^{s-1}d_j^{r_1}+\delta_2\sum_{j=1}^{s-1}d_j^{r_2}+\cdots+\delta_k\sum_{j=1}^{s-1}d_j^{r_k}\right)\|\boldsymbol{e}^{s-j}\|_\infty+C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\leqslant[(\delta_1 d_1^{\gamma_1}+\delta_2 d_1^{\gamma_2}+\cdots+\delta_k d_1^{\gamma_k})\sigma_{s-1}^{-1}+(\delta_1 d_2^{\gamma_1}+\delta_2 d_2^{\gamma_2}+\cdots+\delta_k d_2^{\gamma_k})\sigma_{s-1}^{-1}+\cdots+(\delta_1 d_s^{\gamma_1}+\delta_2 d_s^{\gamma_2}+\cdots+\delta_k d_s^{\gamma_k})\sigma_0^{-1}+1]C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\sigma_s^{-1}。$$

因此$\|\boldsymbol{e}^{s+1}\|_\infty\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\sigma_s^{-1}$,即$\|\boldsymbol{e}^n\|_\infty\leqslant C\Gamma(2-\gamma_1)\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)\sigma_{n-1}^{-1}$ 对 $n=s+1$ 也成立。

下面证明 $\Gamma(2-\gamma_1)\sigma_{n-1}^{-1}$ 是有上界的。由引理 2 和引理 3 得:

$$\Gamma(2-\gamma_1)\sigma_{n-1}^{-1}<\Gamma(2-\gamma_1)\frac{n^{\gamma_1}}{1-\gamma_1}=\Gamma(1-\gamma_1)n^{\gamma_1}。$$

故存在常数 $C>0$,使得$\|\boldsymbol{e}^n\|_\infty\leqslant Cn^{\gamma_1}\tau^{\gamma_1}(\tau^{2-\gamma_1}+h)=C(n\tau)^{\gamma_1}(\tau^{2-\gamma_1}+h)$,$n=1,2,\cdots,N$。 证毕

如果 $n\tau\leqslant T$ 是有限的,则可以得到如下定理。

定理 3 设 u_i^n 是利用加权有限差分格式(9)计算出来的关于 $u(x_i,t_n)$的近似解,于是存在正常数 $\overline{C}=T^{\gamma_1}C$,满足 $|u(x_i,t_n)-u_i^n|\leqslant\overline{C}(\tau^{2-\gamma_1}+h)$,$i=1,2,\cdots,M-1$;$n=1,2,\cdots,N$。

4 数值实例

算例 1 考虑如下空间分数阶对流-扩散方程:

$$\frac{\partial u(x,t)}{\partial t}=-\frac{\partial u(x,t)}{\partial x}+a(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}+b(x)\frac{\partial^\beta u(x,t)}{\partial x^\beta}+f(x,t),0\leqslant x\leqslant 1,0\leqslant t\leqslant 1;$$

$$u(x,0)=x^3(1-x),0\leqslant x\leqslant 1;u(0,t)=0,u(1,t)=0,0\leqslant t\leqslant 1。$$

其中:$\alpha=1.6$,$\beta=0.6$,$a(x)=\Gamma(2.3)x^{0.6}$,$b(x)=\Gamma(0.2)x^{0.8}$,$f(x,t)=(-2x^2+13x-3)\mathrm{e}^{-t}$。上述方程的精确解为 $u(x,t)=x^3(1-x)\mathrm{e}^{-t}$。

令 $\bar{s}=\left\{\frac{1}{h}+\alpha\frac{a_{\max}}{h^\alpha}+\beta\frac{b_{\max}}{h^\beta}\right\}\tau$,则有 $\tau=\dfrac{\bar{s}}{\left\{\frac{1}{h}+\alpha\frac{a_{\max}}{h^\alpha}+\beta\frac{b_{\max}}{h^\beta}\right\}}=\dfrac{\bar{s}}{\left\{\frac{1}{h}+1.6\frac{a_{\max}}{h^{1.6}}+0.6\frac{b_{\max}}{h^{0.6}}\right\}}$。由最大模范数 $E_\infty=\max\limits_{i\in[1,N-1]}|U(x_i,T)-u(x_i,T)|$,取 $h=\frac{1}{200}$和 $\bar{s}=2$,τ 由上式给出,此时定理中的稳定性条件是满足的,计算得到 $E_\infty=0.253\times 10^{-3}$。定理给出的收敛阶为 $O(\tau^2+h)$。选 2 组数(τ_1,h_1)和(τ_2,h_2)满足 $h_1/h_2=2$ 且 $\tau_1^2/\tau_2^2=2$ 来验证收敛阶,由此得到 $\tau_2=\tau_1\times 0.5^{1/2}$。取 $r=0.5^{1/2}$,表 1 给出了在 $T=1$ 时的结果,算例 1 的误差

比率是由前一行的 E_∞ 与该行的 E_∞ 的值相比得到的。可以看出，它们的值近似等于 2，从而验证了定理的结论。

算例 2　考虑如下三项时间-双边空间分数阶对流-扩散方程：

$$\sum_{l=1}^{3}\theta_l \frac{\partial^{\gamma_l}u(x,t)}{\partial t^{\gamma_l}}=-\frac{\partial u(x,t)}{\partial x}+a_+(x,t)\frac{\partial^{1.6}u(x,t)}{\partial_+ x^\alpha}+a_-(x,t)\frac{\partial^{1.6}u(x,t)}{\partial_- x^\alpha}+$$

$$b_+(x,t)\frac{\partial^{1.8}u(x,t)}{\partial_+ x^\alpha}+b_-(x,t)\frac{\partial^{1.8}u(x,t)}{\partial_- x^\alpha}+f(x,t),0\leqslant x\leqslant 1,0\leqslant t\leqslant 1;$$

$$u(x,0)=x^3(1-x),0\leqslant x\leqslant 1;u(0,t)=0,u(1,t)=0,0\leqslant t\leqslant 1。$$

式中：

$$\gamma_1=0.8,\gamma_2=0.6,\gamma_3=0.6,\theta_1=\theta_2=\theta_3=\frac{1}{3},a_+(x,t)=\Gamma(0.4)x^{0.6}t^3,$$

$$d_-(x,t)=\Gamma(0.4)(1-x)^{1.6}\times(1+2t^3),b_+(x,t)=\Gamma(0.2)x^{0.8}t^3,b_-(x,t)=\Gamma(0.2)(1-x)^{1.8}(1+2t^3),$$

$$f(x,t)=\frac{x^3t^{2.4}}{\Gamma(3.4)}+\frac{x^3t^{2.2}}{\Gamma(3.2)}+(1+2t^3)(-25x^3+40x^2-12x)。$$

上述方程的精确解为 $u(x,t)=x^3(1-x)(1+2t^3)$。

取定时间步长 $\tau=0.000\ 1$，空间步长 $h=0.02$。由定理 3 可知误差 $\|\boldsymbol{e}\|_\infty\leqslant\overline{C}(\tau^{2-\gamma_1}+h)$，此处 $\gamma_1=0.8$，因此需要令 $r=0.5^{5/6}$ 来检验一下收敛阶。表 2 给出了在 $T=1$ 时的最大误差和误差比率。可以看出此误差比率与 2 很接近，这就验证了加权隐式差分格式(10)的收敛阶。

表 1　最大模误差和误差比率，其中 $r=0.5^{1/2}$

Tab. 1　Maximum modulus error and error ratio when $r=0.5^{1/2}$

h	τ	$\boldsymbol{e}_\infty$	比率
0.2	0.001	0.001 546 44	—
0.1	$0.001\times r$	0.000 446 48	1.894 17
0.05	$0.001\times r^2$	0.000 215 646	1.894 66
0.025	$0.001\times r^3$	0.000 481 167	1.946 51
0.012 5	$0.001\times r^4$	9.07E−05	1.961 67
0.006 25	$0.001\times r^5$	5.46E−05	1.976 41

表 2　最大模误差和误差比率，其中 $r=0.5^{6/5}$

Tab. 2　Maximum modulus error and error ratio when $r=0.5^{6/5}$

h	τ	$\boldsymbol{e}_\infty$	比率
0.2	0.005	0.001 326 534	—
0.1	$0.005\times r$	0.000 685 140	1.936 15
0.05	$0.005\times r^2$	0.000 351 669	1.948 25
0.025	$0.005\times r^3$	0.000 179 662	1.957 40
0.012 5	$0.005\times r^4$	9.077 44E−05	1.979 21
0.006 25	$0.005\times r^5$	4.564 76E−05	1.988 59

进一步比较算例 1 和算例 2，加权隐式差分格式对应的数值解与精确解的比较结果如图 1 和表 3 所示，从表中和图中可以观察出，加权隐式差分格式的近似解与精确解都很接近，但对三项时间-双边空间分数阶对流-扩散方程的加权隐式差分结果比普通空间分数阶对流-扩散方程的精确度要稍显高些。

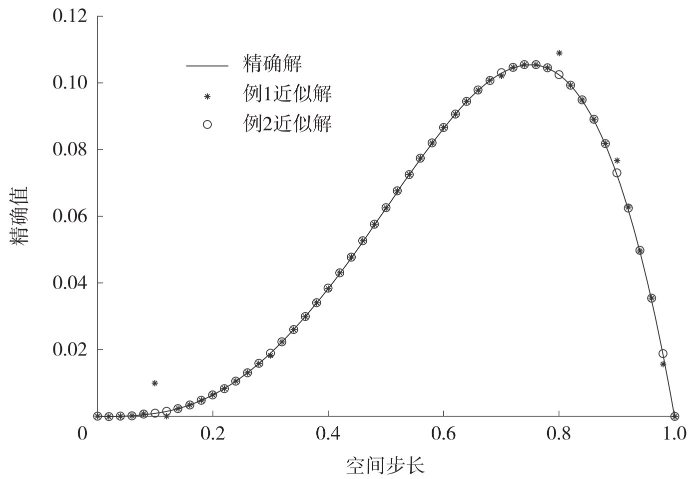

图 1　近似解与精确解比较图

Fig. 1　Comparison between the numerical solution and exact solution

表 3 近似解与精确解对照表

Tab. 3 Comparison table for numerical solution and exact solution

精确解	算例 1 近似解	误差	算例 2 近似解	误差	精确解	算例 1 近似解	误差	算例 2 近似解	误差
0	0	—	0	—	0.067 492	0.067 492	0	0.067 492	0
0.000 008	0.000 008	0.000 526	0.000 008	0.000 301	0.072 433	0.072 433	0	0.072 433	0
0.000 061	0.000 061	0.000 074	0.000 061	0.000 154	0.077 271	0.077 271	0	0.077 271	0
0.000 203	0.000 203	0.000 011	0.000 203	0.000 003	0.081 947	0.081 947	0	0.081 947	0
0.000 471	0.000 773	0.641 351	0.000 471	0.000 011	0.086 400	0.086 657	0.002 970	0.086 477	0.000 892
0.000 900	0.009 932	0.036 018	0.000 923	0.025 500	0.090 565	0.090 565	0.000 001	0.090 565	0
0.001 521	0.000 152	0.899 829	0.001 521	0.000 004	0.094 372	0.094 372	0	0.094 372	0
0.002 360	0.002 360	0.000 004	0.002 360	0.000 001	0.097 749	0.097 746	0.000 031	0.097 749	0
0.003 441	0.003 441	0	0.003 441	0.000 002	0.100 618	0.100 619	0.000 007	0.100 618	0
0.004 782	0.004 783	0.000 204	0.004 782	0.000 002	0.102 900	0.102 126	0.007 525	0.102 951	0.000 495
0.006 400	0.006 492	0.014 321	0.006 451	0.007 960	0.104 509	0.104 555	0.000 432	0.104 509	0
0.008 305	0.008 305	0.000 001	0.008 305	0.000 001	0.105 358	0.105 357	0.000 016	0.105 358	0
0.010 506	0.010 532	0.002 491	0.010 506	0.000 001	0.105 354	0.105 386	0.000 298	0.105 354	0
0.013 006	0.013 006	0	0.013 006	0	0.104 401	0.104 423	0.000 208	0.104 401	0
0.015 805	0.015 805	0	0.015 805	0	0.102 400	0.108 866	0.063 141	0.102 400	0
0.018 900	0.018 324	0.030 469	0.018 900	0.000 001	0.099 246	0.099 257	0.000 105	0.099 246	0
0.022 282	0.022 282	0	0.022 282	0	0.094 833	0.094 823	0.000 098	0.094 833	0
0.025 941	0.025 941	0	0.025 941	0	0.089 048	0.089 048	0	0.089 048	0
0.029 860	0.029 860	0	0.029 860	0	0.081 777	0.081 777	0	0.081 777	0
0.034 021	0.034 020	0.000 012	0.034 021	0	0.072 900	0.076 566	0.050 290	0.072 924	0.000 324
0.038 400	0.038 320	0.002 073	0.038 482	0.002 134	0.062 295	0.062 653	0.005 753	0.062 295	0
0.042 971	0.042 971	0	0.042 971	0	0.049 835	0.049 567	0.005 387	0.049 835	0
0.047 703	0.047 703	0	0.047 703	0	0.035 389	0.035 389	0	0.035 389	0
0.052 561	0.052 561	0.000 003	0.052 561	0	0.018 824	0.015 646	0.168 838	0.018 824	0
0.057 508	0.057 598	0.001 576	0.057 508	0	0	0	—	0	—
0.062 500	0.062 403	0.001 547	0.062 582	0.001 308					

5 总结

多项时间-两边空间分数阶对流-扩散方程的初边值问题的数值解，是有限差分法研究的一类数学模型，本文基于移位 Grünwald-Letnikov 公式，将方程中的空间分数阶导数采用加权平均有限差分法近似，得到了一种加权隐式有限差分格式。利用能量估计，得到了该差分格式的稳定性。然后利用数学归纳法证明了在相同的条件下，所提出的差分格式是收敛的。最后通过 2 个数值算例验证了文中的加权隐式差分格式的收敛阶，另外验证了加权隐式差分格式的近似解与精确解都很接近，特别是三项时间-双边空间分数阶对流-扩散方程的加权隐式差分结果比普通空间分数阶对流-扩散方程的精确度要稍显高些。说明本文所提出的差分格式是可靠和有效的。

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A Weighted Implicit Difference Scheme for Multi-Term Time Fractional Advection-Diffusion Equation with Two-Sided Space Fractional Derivatives

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Abstract: A kind of multiple-time two-sided space fractional advection-diffusion equation is considered. Based on the shifted Grünwald-Letnikov formula, the spatial fractional order derivatives in the equations are approximated by the weighted average finite difference method, energy estimation is used, mathematical induction and numerical examples are used to illustrate the reliability and validity of the proposed difference format. A weighted implicit finite difference format, the stability of the difference format, and the convergence of the difference format are achieved. By comparing the numerical and exact solutions of the example equation, the theoretical results are verified.

Keywords: fractional advection diffusion equation; space fractional derivative; weighted implicit scheme; convergence; stability; finite difference method

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