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Optimality Conditions and Duality for Nondifferentiable Multiobjective Programming Problems with (C,α,ρ,d)-Convexity*

LONG Xian-jun, HUANG Ying-quan

(Math. & Statistics College, Chongqing Technology and Business University, Chongqing 400067, China)

Abstract: In this paper, we consider the following nondifferentiable multiobjective programming problem: (MP) min $(f_1(x)+s(x\mid C_1), f_2(x)+s(x\mid C_2), \cdots, f_p(x)+s(x\mid C_p))$, s. t. $h(x)\leqslant 0$, where $f_i: X\to\mathbf{R}$, $i=1,2,\cdots,p$ and $h=(h_1,h_2,\cdots,h_m): X\to\mathbf{R}^m$, are continuously differentiable functions over X; C_i, for each $i\in\{1,2,\cdots,p\}$, is a compact convex set of $\mathbf{R}^n$, and $s(x\mid C_i)$ denotes the support function of C_i evaluated at x. Under the assumption of (C,α,ρ,d)-convexity, the Kuhn-Tucher type sufficient optimality conditions for weakly efficient solutions of the nondifferentiable multiobjective programming problem are established. Moreover, the Mond-Weir type dual model is formulated nad duality theorems are obtained. Our results generalize some recent results in the literature.

Key words: nondifferentiable multiobjective programming; optimality condition; duality; weakly efficient solution; (C,α,ρ,d)-convexity

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1 Introduction

Multiobjective programming has been extensivley studied over the past few decades due to it has many applications in such fields as the Internet, finance, biomedicine, management science, game theory and engineering. A large number of results have appeared in the literature[1-5].

As is well-known, convexity plays an important role in the design and analysis of successful algorithms for solving optimization problems. However, the condition of convexity is too strong. Therefore, several classes of generalized convex functions have been introduced in the literature, such as invexity[7], (V,ρ)-invexity[3], (F,ρ)-convexity[5], F-convexity[8], ρ-convexity[9], (F,α,ρ,d)-convexity[10]. Recently, Yuan[11] introduced a class of functions, which called (C,α,ρ,d)-convex function and which includes (F,α,ρ,d)-convexity[10], V-ρ-invexity[3], (F,ρ)-convexity[5] as special cases. Therefore, it is important to research the optimization conditions and duality results for multiobjective programming problems under conditions of (C,α,ρ,d)-convexity.

In a recent paper[12], Mond and Schechter studied non-differentiable symmetric duality, in which the objective functions contain a support function. Based on the ideas of Mond and Schechter[12], Yang[6] studied generalized dual problems for a class of nondifferentiable multiobjective programs.

Inspired and motivated by[6,11], in this paper, we study a class of nondifferentiable multiobjective programming problems in which each component of the objective function contains a term involving the support function of a compact convex set. We obtain soms sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective programming problems under the assumptions of (C,α,ρ,d)-convexity.

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作者简介:龙宪军,男,讲师,博士,研究方向为优化理论及应用。

2 Preliminaries

Throughout this paper, let $\mathbf{R}^n$ be the n-dimensional Euclidean space and $\mathbf{R}^n_+$ be nonnegative orthant of $\mathbf{R}^n$. Let X be an open subset of $\mathbf{R}^n$. Assume that $\alpha: X\times X\to\mathbf{R}+\backslash\{0\}$, $\rho\in\mathbf{R}$ and $d: X\times X\to\mathbf{R}_+$ satisfies $d(x,x_0)=0\Leftrightarrow x=x_0$. Let $C: X\times X\times\mathbf{R}^n\to\mathbf{R}$ be a function satisfies $C_{(x,x_0)}(0)=0$ for any $(x,x_0)\in X\times X$.

Definition 1 A function $C: X\times X\times\mathbf{R}^n\to\mathbf{R}$ is said to be convex on $\mathbf{R}^n$ if for any fixed $(x,x_0)\in X\times X$ and for any $y_1,y_2\in\mathbf{R}^n$, one has

$$C_{(x,x_0)}(\lambda y_1+(1-\lambda)y_2)\leqslant\lambda C_{(x,x_0)}(y_1)+(1-\lambda)C_{(x,x_0)}(y_2),\ \forall\lambda\in(0,1)$$

Definition 2[12] A differentiable function $h: X\to\mathbf{R}$ is said to be (C,α,ρ,d)- convex at $x_0\in X$ if for any $x\in X$, $\dfrac{h(x)-h(x_0)}{\alpha(x,x_0)}\geqslant C_{(x,x_0)}(\nabla h(x_0))+\rho\dfrac{d(x,x_0)}{\alpha(x,x_0)}$.

The function h is said to be (C,α,ρ,d)-convex on X if it is (C,α,ρ,d)-convex at every point in X. In particular, h is said to be strongly (C,α,ρ,d)-convex on X if $\rho>0$.

Remark 1 If the function C is sublinear with respect to the third argument. then the (C,α,ρ,d)-convexity is the same as the (F,α,ρ,d)-convexity introduced by Liang[6].

Remark 2 Every (F,α,ρ,d)-convex function is (C,α,ρ,d)-convex. However, the converse is not true.

Example 1 Let $X=\{x:\frac{7\pi}{4}\leqslant x\leqslant 2\pi$, $\rho=-1$, $\alpha(x,x_0)=1$, $d(x,x_0)=\sqrt{(x-x_0)^2}$ and $C(x,x_0;a)=a^2(x-x_0)$ for any $(x,x_0)\in X\times X$. Let $h(x)=\cos^2x$. Obviously, the function C is not sublinear with respect to the third argument. Then, h is not (F,α,ρ,d)-convex at $x_0=\frac{7\pi}{4}$. It is easy to prove that h is (C,α,ρ,d)-convex at $x_0=\frac{7\pi}{4}$.

We consider the following multiobjective programming problem

$$\text{(MP)}\quad \min\quad (f_1(x)+s(x|C_1),f_2(x)+s(x|C_2),\cdots,f_p(x)+s(x|C_p))$$
$$\text{s.t.}\quad h(x)\leqslant 0$$

where $f_i: X\to\mathbf{R}$, $i=1,2,\cdots,p$ and $h=(h_1,h_2,\cdots,h_m): X\to\mathbf{R}^m$, are continuously differentiable functions over X. Suppose that C_i, for ecah $i\in\{1,2,\cdots,p\}$, is a compact convex set of $\mathbf{R}^n$, and $s(x|C_i)$ denotes the support function of C_i evaluated at x, defined by $s(x|C_i)=\max\{\langle x,w\rangle|w\in C_i\}$. Let $S=\{x\in X: h(x)\leqslant 0\}$ be the set of all feasible solutions and let $I(x):=\{j: h_j(x)=0\}$ for any $x\in X$.

Let $k_i(x)=s(x|C_i)$, $i=1,2,\cdots,p$. Then, k_i is a convex function and $\partial k_i(x)=\{w\in C_i|\langle w,x\rangle=s(x|C_i)\}$, where ∂k_i is the subdifferentiable of k_i[7].

3 Optimality Conditions

In this section, we obtain some sufficient optimality conditions for a weakly efficient solutions of (MP) under the assumption of (C,α,ρ,d)-convexity.

Theorem 1 Let $x_0\in S$ be a feasible solution of (MP). Assume that there exist $\lambda_i>0$, $i=1,2,\cdots,p$, and $\mu_j\geqslant 0$, $j=1,2,\cdots,m$, such that

$$\sum_{i=1}^{p}\lambda_i\nabla[f_i(x_0)+\langle w_i,x_0\rangle]+\sum_{j=1}^{m}\mu_j\nabla h_j(x_0)=0\qquad(1)$$

$$\langle w_i,x_0\rangle=s(x_0\mid C_i),w_i\in C_i,i=1,2,\cdots,p\qquad(2)$$

$$\sum_{j=1}^{m}\mu_jh_j(x_0)=0\qquad(3)$$

If $f_i(\cdot)+\langle w_i,\cdot\rangle$, $(i=1,2,\cdots,p)$, is (C,α_i,ρ_i,d_i)-convex at x_0, $h_j(\cdot)$, $j=1,2,\cdots,m$, is $(C,\beta_j,\eta_j,$

c_j)-convex at x_0, and

$$\sum_{i=1}^{p}\lambda_i\rho_i\frac{d_i(x,x_0)}{\alpha_i(x,x_0)}+\sum_{j=1}^{m}\mu_j\eta_j\frac{c_j(x,x_0)}{\beta_j(x,x_0)}\geqslant 0 \qquad (4)$$

then x_0 is a weakly efficient solution of (MP).

Proof. Suppose that x_0 is not a weakly efficient solution of (MP). Then, there existx $x\in S$ such that

$$f_i(x)+s(x\mid C_i)<f_i(x_0)+s(x_0\mid C_i),i=1,2,\cdots,p$$

By(2) and $\langle w_i,x\rangle\leqslant s(x\mid C_i)$ for i = 1,2,…,p, one has

$$f_i(x)+\langle w_i,x\rangle\leqslant f_i(x)+s(x\mid C_i)<f_i(x_0)+s(x_0\mid C_i)=f_i(x_0)+\langle w_i,x_0\rangle \qquad (5)$$

Since $f_i(\cdot)+\langle w_i,\cdot\rangle,(i=1,2,\cdots,p)$,is (C,α_i,ρ_i,d_i)-convex at x_0,i. e. ,

$$\frac{[f_i(x)+\langle w_i,x\rangle-(f_i(x_0)+\langle w_i,x_0\rangle)]}{\alpha_i(x,x_0)}\geqslant C_{(x,x_0)}(\nabla[f_i(x_0)+\langle w_i,x_0\rangle])+\rho_i\frac{d_i(x,x_0)}{\alpha_i(x,x_0)} \qquad (6)$$

By the (C,β_j,η_j,c_j)-convexity of $h_j(\cdot)(j=1,2,\cdots,m)$,one has

$$\frac{h_j(x)-h_j(x_0)}{\beta_j(x,x_0)}\geqslant C_{(x,x_0)}(\nabla h_j(x_0))+\eta_j\frac{c_j(x,x_0)}{\beta_j(x,x_0)} \qquad (7)$$

Denote $\tau=\sum_{i=1}^{p}\lambda_i+\sum_{j=1}^{m}\mu_j$. It is easy to see that $\tau>0$. Multiplying both side of (6) by $\frac{\lambda_i}{\tau}$ and of (7) by $\frac{\mu_j}{\tau}$, respectively, and adding them and using the convexity of $C_{(x,x_0)}(\cdot)$, we get

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau\,\alpha_i(x,x_0)}[f_i(x)+\langle w_i,x\rangle-(f_i(x_0)+\langle w_i,x_0\rangle)]+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\frac{h_j(x)-h_j(x_0)}{\beta_j(x,x_0)}\geqslant$$

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau}(C_{(x,x_0)}(\nabla[f_i(x_0)+\langle w,x_0\rangle]))+\sum_{j=1}^{m}\frac{\mu_j}{\tau}C_{(x,x_0)}(\nabla h_j(x_0))+$$

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau}\rho_i\frac{d_i(x,x_0)}{\alpha_i(x,x_0)}+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\eta_j\frac{c_j(x,x_0)}{\beta_j(x,x_0)}\geqslant C_{(x,x_0)}(\frac{1}{\tau}[\sum_{i=1}^{p}\lambda_i\nabla(f_i(x_0)+\langle w_i,x_0\rangle)+\sum_{j=1}^{m}\mu_j\nabla h_j(x_0)])+$$

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau}\rho_i\frac{d_i(x,x_0)}{\alpha_i(x,x_0)}+\sum_{j=1}^{m}\frac{\mu_i}{\tau}\eta_j\frac{c_j(x,x_0)}{\beta_j(x,x_0)}$$

This fact together with (1) and (4) yields

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau\,\alpha_i(x,x_0)}[f_i(x)+\langle w_i,x\rangle-(f_i(x_0)+\langle w_i,x_0\rangle)]+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\frac{h_j(x)-h_j(x_0)}{\beta_j(x,x_0)}\geqslant 0 \qquad (8)$$

Since x_0 is a feasible solution of (MF), it follows from (3) that

$$\sum_{j=1}^{m}\mu_j\frac{h_j(x)-h_j(x_0)}{\beta_j(x,x_0)}\leqslant 0 \qquad (9)$$

Combining (5) and (9) yields

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau\,\alpha_i(x,x_0)}[f_i(x)+\langle w_i,x\rangle-(f_i(x_0)+\langle w_i,x_0\rangle)]+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\frac{h_j(x)-h_j(x_0)}{\beta_j(x,x_0)}<0,$$

which contradicts to (8). Therefore, x_0 is a weakly efficient solution of (MP).

Corollary 1　Let $x_0\in S$ be a feasible solution of (MP). Assume that there exist $\lambda_i>0,i=1,2,\cdots,p$ and $\mu_j\geqslant 0,j=1,2,\cdots,m$, such that

$$\sum_{i=1}^{p}\lambda_i\nabla[f_i(x_0)+\langle w_i,x_0\rangle]+\sum_{j=1}^{m}\mu_j\nabla h_j(x_0)=0,$$

$$\langle w_i,x_0\rangle=s(x_0\mid C_i),w_i\in C_i,i=1,2,\cdots,p,$$

$$\sum_{j=1}^{m}\mu_jh_j(x_0)=0.$$

If $f_i(\cdot)+\langle w_i,\cdot\rangle,(i=1,2,\cdots,p)$, is strongly (C,α_i,ρ_i,d_i)-convex at x_0, $h_j(\cdot),(j=1,2,\cdots,m)$, is strongly (C,β_j,η_j,c_j)-convex at x_0, then x_0 is a weakly efficient solution of (MP).

Proof. We can easily check that (4) holds under the assumptions of the corollary.

4 Duality Results

In this section, we consider the following Mond-Weir type dual (MD) to the primal problem (MP)

$$\begin{cases} \max & (f_1(u)+\langle w_1,u\rangle,\cdots,f_p(u)+\langle w_p,u\rangle) \\ \text{s. t.} & \sum_{i=1}^{p}\lambda_i\nabla[f_j(u)+\langle w_i,u\rangle]+\sum_{j=1}^{m}\mu_j\nabla h_j(u)=0 \\ & \sum_{j=1}^{m}\mu_j h_j(u)\geq 0, \\ & w:=(w_1,w_2,\cdots,w_p),w_i\in C_i,i=1,2,\cdots,p,u\in X \\ & \mu_j\geq 0,j=1,2,\cdots,m,\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_p)\in\Lambda^+ \end{cases} \tag{10}$$

where

$$\Lambda^+=\{\lambda\in R_+^p:\lambda_i>0\}$$

Theorem 2 (Weak Duality) Let x and (u,λ,w,μ) be the feasible solutions of (MP) and (MD), respectively. Assume that $f_i(\cdot)+\langle w_i,\cdot\rangle,(i=1,2,\cdots,p)$, is (C,α_i,ρ_i,d_i)-convex at u, and $h_j(\cdot),(j=1,2,\cdots,m)$, is (C,β_j,η_j,c_j)-convex at u. If

$$\sum_{i=1}^{p}\lambda_i\rho_i\frac{d_i(x,u)}{\alpha_i(x,u)}+\sum_{j=1}^{m}\mu_j\eta_j\frac{c_j(x,u)}{\beta_j(x,u)}\geq 0 \tag{11}$$

then the following cannot hold

$$(f_1(x)+s(x\mid C_1),\cdots,f_p(x)+s(x\mid C_p))<(f_1(u)+\langle w_1,u\rangle,\cdots,f_p(u)+\langle w_p,u\rangle) \tag{12}$$

Proof. Let x and (u,λ,w,μ) be the feasible solutions of (MP) and (MD), respectively. It follows that

$$\sum_{j=1}^{m}\mu_j h_j(x)\leq 0\leq\sum_{j=1}^{m}\mu_j h_j(u)$$

Since $h_j(\cdot),(j=1,2,\cdots,m)$, is (C,β_j,η_j,c_j)-convex at u, one has

$$0\geq\sum_{j=1}^{m}\mu_j\frac{h_j(x)-h_j(u)}{\beta_j(x,u)}\geq\sum_{j=1}^{m}\mu_j C_{(x,u)}(\nabla h_j(u))+\sum_{j=1}^{m}\mu_j\eta_j\frac{c_j(x,u)}{\beta_j(x,u)} \tag{13}$$

Now suppose, contrary to the results, that (12) holds. This together with $s(x\mid C_i)\geq\langle w_i,x\rangle$, $i=1,2,\cdots,p$, gives that

$$f_i(x)+\langle w_i,x\rangle\leq f_i(x)+s(x\mid C_i)<f_i(u)+\langle w_i,u\rangle \tag{14}$$

By the (C,α_i,ρ_i,d_i)-convexity of $f_i(\cdot)+\langle w_i,\cdot\rangle,i=1,2,\cdots,p$

$$\frac{[f_i(x)+\langle w_i,x\rangle-(f_i(u)+\langle w_i,u\rangle)]}{\alpha_i(x,u)}\geq C_{(x,u)}(\nabla[f_i(u)+\langle w_i,u\rangle])+\rho_i\frac{d_i(x,u)}{\alpha_i(x,u)} \tag{15}$$

Denote $\tau=\sum_{i=1}^{p}\lambda_i+\sum_{j=1}^{m}\mu_j$. It follows from (10)—(11), (13)—(15) and the convexity of $C_{(x,u)}(\cdot)$ that

$$0>\sum_{i=1}^{p}\frac{\lambda_j}{\tau\,\alpha_i(x,u)}[f_i(x)+\langle w_i,x\rangle-(f_i(u)+\langle w_i,u\rangle)]+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\frac{h_j(x)-h_j(u)}{\beta_j(x,u)}\geq$$

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau}(C_{(x,u)}(\nabla[f_i(u)+\langle w,u\rangle]))+\sum_{j=1}^{m}\frac{\mu_j}{\tau}C_{(x,u)}(\nabla h_j(u))+$$

$$\sum_{i=1}^{p}\frac{\lambda_j}{\tau}\rho_i\frac{d_i(x,u)}{\alpha_i(x,u)}+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\eta_j\frac{c_j(x,u)}{\beta_j(x,u)}\geq$$

$$C_{(x,u)}(\frac{1}{\tau}[\sum_{i=1}^{p}\lambda_i\nabla(f_i(u)+\langle w_i,u\rangle)+\sum_{j=1}^{m}\mu_j\nabla h_j(u)])+$$

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau}\rho_i\frac{d_i(x,u)}{\alpha_i(x,u)}+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\eta_j\frac{c_j(x,u)}{\beta_j(x,u)}\geq 0$$

which gives a contradiction. This completes the proof.

Corollary 2 (Weak Duality) Let x and (u,λ,w,μ) be the feasible solutions of (MP) and (MD), respectively. Assume that $f_i(\cdot) + \langle w_i, \cdot\rangle$, $(i = 1,2,\cdots,p)$, is strongly (C,α_i,ρ_i,d_i)-convex at u, and $h_j(\cdot)$, $(j = 1,2,\cdots,m)$, is strongly (C,β_j,η_j,c_j)-convex at u. Then the following cannot hold

$$(f_1(x) + s(x \mid C_1),\cdots,f_p(x) + s(x \mid C_p)) < (f_1(u) + \langle w_1,u\rangle,\cdots,f_p(u) + \langle w_p,u\rangle)$$

Proof. We can easily check that (11) holds under the assumptions of the corollary.

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运筹学与控制论

不可微多目标规划问题的最优性条件和对偶

龙宪军，黄应全

（重庆工商大学 数学与统计学院，重庆 400067）

摘要 研究了如下的不可微多目标规划问题：(MP) $\min\ (f_1(x) + s(x \mid C_1), f_2(x) + s(x \mid C_2), \cdots, f_p(x) + s(x \mid C_p))$，s. t. $h(x) \leqslant 0$，其中函数 $f_i: X \to \mathbf{R}$，$(i = 1,2,\cdots,p)$ 和 $h = (h_1,h_2,\cdots,h_m): X \to \mathbf{R}^m$ 在 X 上是连续可微的；$C_i\ (i \in \{1,2,\cdots,p\})$ 是 $\mathbf{R}^n$ 上的紧凸集，$s(x \mid C_i)$ 表示集合 C_i 在 x 的支撑函数。在 (C,α,ρ,d)-凸性的假设下，得到了不可微多目标规划问题弱有效解的 Kuhn-Tucher 型最优性充分条件。而且本文得到了原问题的 Mond-Weir 型对偶以及相应的对偶结果。本文所得结果推广了一些最新的结果。

关键词 不可微多目标规划问题；最优性条件；对偶；弱有效解；(C,α,ρ,d)-凸性

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