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A Characterization for *r*-Preinvex Function*

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Abstract: In this paper, an equivalent condition for a class of *r*-preinvex function is established. A characterization for a twice continuously dierentiable *r*-preinvex function is obtained by the equivalent condition. Under some suitable conditions, the following result has been proved: Let $X\subseteq \mathbf{R}^n$ be open invex set with respect to $\eta: X\times X\to \mathbf{R}^n$ and η satisfy condition C. f defined on X is twice continuously dierentiable and satisfies condition D. Then f is *r*-preinvex function with respect to η if and only if $\forall x, y\in X, r[\nabla f(y)^{\mathrm{T}}(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant 0$. Our results improve and generalize some known results.

Key words: invex set; *r*-convex function; preinvex function; *r*-preinvex function
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1 Introduction

It is well known that convexity has been playing an central role in mathematical programming, engineering and optimization theory. Avriel[1] introduced the concept of *r*-convex function which is a generalization of convex function and discussed some characterizations of *r*-convex function. Ben-Israel and Mond[2] considered function (not necessarily differentiable) for which there exists a vector function $\eta: \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ such that, for any $x, y\in\mathbf{R}^n$, $\lambda\in[0,1]$

$$f(y+\lambda\eta(x,y))\leqslant \lambda f(x)+(1-\lambda)f(y) \tag{1}$$

Weir, et al[3-4] named this kind of function which was satisfied (1) preinvex function with respect to η. As the generalizations of *r*-convex function and preinvex function, Antczak[5] gave the definition of *r*-preinvex function and obtained some optimality results under *r*-preinvexity assumption for a class of constrained optimization problems.

Motivated by works of Avriel[1] and Antczak[5], in this paper, under some suitable conditions, we first give an equivalent condition for *r*-preinvex function as follows: f is *r*-preinvex function with respect to η if and only if $F(x, y,\lambda)$ is *r*-convex function with respect to λ in $[0,1]$. Based on the result, we establish a criteria for *r*-preinvex function as follows: f is *r*-preinvex function with respect to η if and only if $\forall x, y\in X$, $r[\nabla f(y)^{\mathrm{T}}\eta(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant 0$. By the criteria, we can verify the *r*-preinvexity of functions.

It is very important and meaningful to study the characterizations of *r*-preinvexity which was introduced bu Antczak in [5] as a generalization of the preinvexity. On the one hand, this will be helpful to understand the nature of *r*-preinvexity. On the other hand, it will also be lay a foundation to study the applications in optimization theory for *r*-preinvexity. In this paper, we give some necessary and suffcient conditions of *r*-preinvexity. This will provide some criteria to verify the *r*-preinvexity, our main results generalize and improvesome corresponding results in [1].

2 Preliminaries

Definition 1[1] Let $f: X\to\mathbf{R}$, where X is a nonempty convex set in $\mathbf{R}^n$. f is said to be *r*-convex on X if $\forall x$,

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$y\in X$, $\forall\lambda\in[0,1]$, $f(y+\lambda(x-y))\leqslant\begin{cases}\log(\lambda e^{rf(x)}+(1-\lambda)e^{rf(y)})^{\frac{1}{r}}, r\neq 0\\ \lambda f(x)+(1-\lambda)f(y), r=0\end{cases}$.

Definition 2[3-4] A nonempty set $X\subseteq\mathbf{R}^n$ is said to be invex if there exists a vector-valued function $\eta: X\times X\to\mathbf{R}^n$ such that $\forall x, y\in X, \forall\lambda\in[0,1]$, $y+\lambda\eta(x,y)\in X$.

Definition 3[3-4] Let $X\subseteq\mathbf{R}^n$ be invex with respect to vector-valued function $\eta: X\times X\to\mathbf{R}^n$ and $f: X\to\mathbf{R}$. f is said to be preinvex with respect to η if $\forall x, y\in X$, $\forall\lambda\in[0,1]$, $f(y+\lambda\eta(x,y))\leqslant\lambda f(x)+(1-\lambda)f(y)$.

Definition 4[5] Let $X\subseteq\mathbf{R}^n$ be invex with respect to $\eta: X\times X\to\mathbf{R}^n$. $f: X\to\mathbf{R}$ is said to be r-preinvex if $\forall x$, $y\in X, \forall\lambda\in[0,1]$, $f(y+\lambda\eta(x,y))\leqslant\begin{cases}\log(\lambda e^{rf(x)}+(1-\lambda)e^{rf(y)})^{\frac{1}{r}}, r\neq 0\\ \lambda f(x)+(1-\lambda)f(y), r=0\end{cases}$.

Condition C[6] Let $\eta: X\times X\to\mathbf{R}^n$ and satisfy condition C if $\forall x, y\in X$, $\forall\lambda\in[0,1]$, $C_1: \eta(y, y+\lambda\eta(x,y))=-\lambda\eta(x,y)$, $C_2: \eta(x, y+\lambda\eta(x,y))=(1-\lambda)\eta(x,y)$.

Condition D[7] Let $X\subseteq\mathbf{R}^n$ be invex with respect to $\eta: X\times X\to\mathbf{R}^n$ and $f: X\to\mathbf{R}$. f is said to be satisfied condition D if $\forall x, y\in X$, $f(y+\eta(x,y))\leqslant f(x)$.

Lemma 1[1] Let F be a twice continuously differentiable real function on an open interval (a,b). Denote by F', F'' the first and second derivatives of F, respectively. Then F is r-convex if and only if for every $x\in(a,b)$, $r[F'(x)]^2+F''(x)\geqslant 0$.

3 Main Results

Avriel[1] established the following result for r-convex function as follows.

Suppose that $X\subseteq\mathbf{R}^n$ be a nonmepty convex set and $f: X\to\mathbf{R}$. $\forall x, y\in X$, $\forall\lambda\in[0,1]$, let $F(x,y,\lambda)=f(y+\lambda(x-y))$. Then f is r-convex function if and only if $F(x,y,\lambda)$ is r-convex function with respect to λ in $[0,1]$.

In the following, we give a generalization of above theorem and establish the equivalent condition of r-preinvex function.

Theorem 1 Let $X\subseteq\mathbf{R}^n$ be invex set with respect to $\eta: X\times X\to\mathbf{R}^n$, η satisfy condition C and f defined on X satisfy condition D. $\forall x, y\in X$, $\forall\lambda\in[0,1]$, let $F(x,y,\lambda)=f(y+\lambda\eta(x,y))$. Then f is r-preinvex function with respect to η if and only if $F(x,y,\lambda)$ is r-convex function with respect to λ in $[0,1]$.

Proof We consider only the case when $r>0$; in other cases, the proof is analogous. Assume that f is r-preinvex function with respect to η, then $\forall x, y\in X$, $\forall\lambda\in[0,1]$

$$f(y+\lambda\eta(x,y))\leqslant\log(\lambda e^{rf(x)}+(1-\lambda)e^{rf(y)})^{\frac{1}{r}} \tag{2}$$

Next, we prove that $F(x,y,\lambda)$ is r-convex function with respect to λ in $[0,1]$. $\forall\alpha_1,\alpha_2\in[0,1]$, $\forall\lambda\in[0,1]$.

If $\alpha_1=\alpha_2$, the result is obvious.

If $\alpha_1>\alpha_2$, then $\alpha_1-\alpha_2>0$ and $\alpha_2\neq 1$. Thus we have $0<\frac{\alpha_1-\alpha_2}{1-\alpha_2}\leqslant 1$.

From the condition C, it follows that

$$\eta(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y))=\eta(y+\alpha_2\eta(x,y)+(\alpha_1-\alpha_2)\eta(x,y), y+\alpha_2\eta(x,y))=$$

$$\eta(y+\alpha_2\eta(x,y)+\frac{\alpha_1-\alpha_2}{1-\alpha_2}\eta(x, y+\alpha_2\eta(x,y)), y+\alpha_2\eta(x,y))=\frac{\alpha_1-\alpha_2}{1-\alpha_2}\eta(x, y+\alpha_2\eta(x,y))=(\alpha_1-\alpha_2)\eta(x,y) \tag{3}$$

Then, by (2) and (3), we obtain

$$F(x,y,\alpha_2+\lambda(\alpha_1-\alpha_2))=f(y+(\alpha_2+\lambda(\alpha_1-\alpha_2))\eta(x,y))=$$

$$f(y+\alpha_2\eta(x,y)+\lambda\eta(y+\alpha_1\eta(x,y),y+\alpha_2\eta(x,y)))\leqslant$$
$$\log(\lambda e^{rf(y+\alpha_1\eta(x,y))}+(1-\lambda)e^{rf(y+\alpha_2\eta(x,y))})^{\frac{1}{r}}=\log(\lambda e^{rF(x,y,\alpha_1)}+(1-\lambda)e^{rF(x,y,\alpha_2)})^{\frac{1}{r}}$$

If $\alpha_1<\alpha_2$, then $\alpha_2-\alpha_1>0$ and $\alpha_1\neq1$. Thus we have $0<\frac{\alpha_2-\alpha_1}{1-\alpha_1}\leqslant1$.

From the condition C, it follows that

$$\eta(y+\alpha_1\eta(x,y),y+\alpha_2\eta(x,y))=\eta(y+\alpha_1\eta(x,y),y+\alpha_1\eta(x,y)+(\alpha_2-\alpha_1)\eta(x,y))=$$
$$\eta(y+\alpha_1\eta(x,y),y+\alpha_1\eta(x,y)+\frac{\alpha_2-\alpha_1}{1-\alpha_1}\eta(x,y+\alpha_1\eta(x,y)))=$$
$$-\frac{\alpha_2-\alpha_1}{1-\alpha_1}\eta(x,y+\alpha_1\eta(x,y))=(\alpha_1-\alpha_2)\eta(x,y) \tag{4}$$

Then, from (2) and (4), we have

$$F(x,y,\alpha_2+\lambda(\alpha_1-\alpha_2))=f(y+(\alpha_2+\lambda(\alpha_1-\alpha_2))\eta(x,y))=$$
$$f(y+\alpha_2\eta(x,y)+\lambda\eta(y+\alpha_1\eta(x,y),y+\alpha_2\eta(x,y)))\leqslant$$
$$\log(\lambda e^{rf(y+\alpha_1\eta(x,y))}+(1-\lambda)e^{rf(y+\alpha_2\eta(x,y))})^{\frac{1}{r}}=\log(\lambda e^{rF(x,y,\alpha_1)}+(1-\lambda)e^{rF(x,y,\alpha_2)})^{\frac{1}{r}}$$

By the definition of r-convex function, $F(x,y,\lambda)$ is r-convex function with respect to λ in $[0,1]$.

Conversely, $\forall x,y\in X$, $\forall\lambda\in[0,1]$, suppose that the $F(x,y,\lambda)$ is r-convex function with respect to λ in $[0,1]$. Then, by the condition D, we can obtain

$$f(y+\lambda\eta(x,y))=F(x,y,\lambda)=F(x,y,\lambda\cdot1+(1-\lambda)\cdot0)\leqslant$$
$$\log(\lambda e^{rF(x,y,1)}+(1-\lambda)e^{rF(x,y,0)})^{\frac{1}{r}}=\log(\lambda e^{rf(y+\eta(x,y))}+(1-\lambda)e^{rf(y)})^{\frac{1}{r}}\leqslant\log(\lambda e^{rf(x)}+(1-\lambda)e^{rf(y)})^{\frac{1}{r}}$$

This completes the proof.

Remark 1 Obviously, let $\eta(x,y)=x-y$, Theorem 1 is concided with the result which was established by Avriel in [1].

In the following, we give a characterization of r-preinvex function and establish an necessary and suffient condition of twice continuously diffierentiable r-preinvex function by making use of Theorem 1.

Theorem 2 Let $X\subseteq\mathbf{R}^n$ be open invex set with respect to $\eta: X\times X\to\mathbf{R}^n$ and η satisfy condition C. f defined on X is twice continuously differentiable and satisfies condition D. Then f is r-preinvex function with respect to η if and only if $\forall x,y\in X$, $r[\nabla f(y)^{\mathrm{T}}\eta(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant0$.

Proof Suppose that f is a twice continuously differentiable r-preinvex function with respect to η. By Theorem 1, $\forall x,y\in X$, $F(x,y,\lambda)=f(y+\lambda\eta(x,y))$ is a twice continuously differentiable r-convex function with respect to λ in $[0,1]$.

From Lemma 1, $\forall\lambda\in(0,1)$, we can obtain that

$$r[F'_\lambda(x,y,\lambda)]^2+F''_\lambda(x,y,\lambda)\geqslant0 \tag{5}$$

Because

$$F'_\lambda(x,y,\lambda)=\eta(x,y)^{\mathrm{T}}\nabla f(y+\lambda\eta(x,y)) \tag{6}$$
$$F''_\lambda(x,y,\lambda)=\eta(x,y)^{\mathrm{T}}\nabla^2 f(y+\lambda\eta(x,y))\eta(x,y) \tag{7}$$

From (5) ~ (7), it follows that

$$r[\nabla f(y+\lambda\eta(x,y))^{\mathrm{T}}\eta(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y+\lambda\eta(x,y))\eta(x,y)\geqslant0 \tag{8}$$

Let $\lambda\to0^+$ in (8), we have $r[\nabla f(y)^{\mathrm{T}}\eta(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant0$

Conversely, assume that, $\forall x,y\in X$, we have $r[\nabla f(y)^{\mathrm{T}}\eta(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant0$.

Then, by the condition C, $\forall\lambda\in(0,1)$, $y+\lambda\eta(x,y)\in X$ and

$$r[\nabla f(y+\lambda\eta(x,y))^{\mathrm{T}}\eta(x,y+\lambda\eta(x,y))]^2+\eta(x,y+\lambda\eta(x,y))^{\mathrm{T}}\nabla^2 f(y+\lambda\eta(x,y))\eta(x,y+\lambda\eta(x,y))\geqslant0$$

Thus, $\forall\lambda\in(0,1)$, from condition C, we have $r[F'_\lambda(x,y,\lambda)]^2+F''_\lambda(x,y,\lambda)\geqslant0$.

Again by Lemma 1, $F(x,y,\lambda)$ is r-convex function with respect to λ in $[0,1]$. By the continuity of f, $F(x,y,\lambda)$ *is* r-convex function with respect to λ in $[0,1]$. Therefore, it follows that f is r-preinvex function with respect to η by Theorem 1. This completes the proof.

Corollary 1 Let $X\subseteq\mathbf{R}^n$ be open invex set with respect to $\eta: X\times X\to\mathbf{R}^n$ and η satisfy condition C. f defined on X is twice continuously differentiable and satisfies condition D. Then f is preinvex function with respect to the vector valued function η if and only if $\forall x,y\in X$, $\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant 0$.

Remark 2 Obviously, Theorem 2 is a generalization of Theorem 5.2 which was established by Avriel in [1].

Remark 3 We can verify r-preinvexity of some functions by making use of Theorem 2.

Example 1 Let $X=(-3,-1)\cup(1,3)$ and X be an open invex set with respect to η, where $\eta(x,y)=\begin{cases}x-y, x,y\in(-3,-1)\\ x-y, x,y\in(1,3)\\ -2-y, x\in(1,3), y\in(-3,-1)\\ 2-y, x\in(-3,-1), y\in(1,3)\end{cases}$. We can verify that η satisfies condition C.

Let $f: X\to\mathbf{R}$ be defined by $f(x)=\begin{cases}(x+2)^2, -3<x<-1\\ (x-2)^2, 1<x<3\end{cases}$.

Obviously, f is twice continuously differentiable and satisfies condition D.

Furthermore, let $r\geqslant-\frac{1}{50}$, $\forall x,y\in X$, we have $r[\nabla f(y)^{\mathrm{T}}\eta(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant 0$. Then, f is r-preinvex function with respect to η $(r\geqslant-\frac{1}{50})$.

Remark 4 It is worth noting that the assumption that η satisfies condition C on X in Theorem 2 is essential. The following example illustrates this point.

Example 2 Let $f(x)=\log(3-\sin x)$, $\eta(x,y)=\frac{1}{2}x-y$. Obviously, $X=(0,\frac{\pi}{2})$ is open invex set with respect to η. But by the fact that $\eta(y, y+\lambda\eta(x,y))\neq-\lambda\eta(x,y)$, holds when $x=\frac{\pi}{3}$, $y=\frac{\pi}{6}$, $\lambda=\frac{1}{2}$, we see that η does not satisfy condition C.

Let $r=1$, $\forall x,y\in X$, we have $r[\nabla f(y)^{\mathrm{T}}\eta(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant 0$.

But by the fact that $f(y+\lambda\eta(x,y))>\log(\lambda e^{rf(x)}+(1-\lambda)e^{rf(y)})^{\frac{1}{r}}$ holds when $x=\frac{3}{5}$, $y=\frac{1}{10}$, $\lambda=\frac{1}{2}$. Then $f(x)$ is not 1-preinvex function with respect to η.

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运筹学与控制论

r-预不变凸函数的一个性质

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摘要：首先建立了一类 r-预不变凸函数的一个等价条件，利用该等价条件给出了二次连续可微的 r-预不变凸函数的一个性质。在适当的假设下，证明了如下结果：设 $X\subseteq \mathbf{R}^n$ 是关于向量值函数的开不变凸集，η 满足条件 C，$f: X\to \mathbf{R}$ 是二次连续可微的函数且满足条件 D。则 f 是关于 η 的 r-预不变凸函数当且仅当对任意的 $\forall x, y\in X$，$r[\nabla f(y)^{\mathrm{T}}(x,y)]^2+\eta(x,y)^{\mathrm{T}}\nabla^2 f(y)\eta(x,y)\geqslant 0$。本文的主要结果推广并改进了一些已有的主要结论。

关键词：不变凸集；r-凸函数；预不变凸函数；r-预不变凸函数

（责任编辑　黄　颖）