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Frequency Domain Analysis for Bifurcation in a Logistic Model with Delay *

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Abstract: In this paper, a class of logistic model with delay is considered. By applying the frequency domain approach and analyzing the associated characteristic equation, the existence of bifurcation parameter point is determined. If the coefficient τ is chosen as a bifurcation parameter, it is found that Hopf bifurcation occurs when the parameter τ passes through a critical value τ_k. The length of delay which preserves the stability of the positive equilibrium is calculated to be between zero and a certain positive constant τ_+. Some numerical simulations show that when the delay τ passes through the critical value τ_0, the positive equilibrium is locally stable and unstable when the delay $\tau > \tau_0$.

Key words: Logistic model; stability; Hopf bifurcation; frequency domain; delay
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0 Introduction

In specifically ecological environment, biological activity of populations is complex and diverse. A large number of studies have shown that delay usually occurs in the biological activity, It is of great concern to biologists that how delay has effect on biological populations. There were a lot of literatures on stability of equilibrium point of Logistic model since Cushing[1] found that a delay could undermine the stability of positive equilibrium and caused periodic oscillations. Cushing[1] investigated the Hopf bifurcation of the following model

$$\frac{dN(t)}{dt} = N(t)\left[\alpha - \beta N(t) - \gamma\left(\int_0^{\infty} K(s)N(t-s)ds\right)^2\right] \quad (1)$$

where $K(s) = \frac{s}{\tau}\exp(\frac{-s}{\tau})$, $\tau > 0$ is a parameter and α, β, γ are all positive constants. Gopalsamy[2] studied the conditions of existence of Hopf bifurcation for the following system

$$\frac{du(t)}{dt} = ru(t)\left[1 - \frac{u(t-h)}{K}\right] \quad (2)$$

and gave an approximation of Hopf bifurcation periodic solutions, where K, r are constants and $h > 0$ is a parameter. Huang and Chen[3] investigated the unconditional stability and Hopf bifurcation of the following Logistic model with delay

$$\frac{dN(t)}{dt} = N(t)\left[\alpha - \beta N(t-\tau) - \gamma\int_{-\infty}^{t} K(t-s)N(s)ds\right] \quad (3)$$

where α, β, γ are positive constants, τ is nonnegative delay parameter and $K(t) = te^{-t}$. In this paper, we will continue to study the local Hopf bifurcation for system (3). It is worth pointing out that the aforementioned work (see [1-3]) is used the state-space formulation for delayed differential

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equations, known as the "time domain" approach[4]. Yet in this paper, we will use the frequency domain approach which was initiated and developed by Allwright[5], Mees and Chua[6], and Moiola and Chen[7-8]. This methodology has some advantages over the classical time-domain methods. A typical one is that it utilizes advanced computer graphical capabilities to bypass quite a lot of profound and difficult mathematical analysis.

In this paper, we will devote our attention to finding the Hopf bifurcation point of model (3). Meanwhile, the length of delay preserving the stability of the positive equilibrium is estimated. The main methodology of study is the frequencydomain approach. To the best of our knowledge, there are few papers[9-10] that deal with the research of Hopf bifurcation by the frequency-domain approach.

The remainder of the paper is organized as follows: in Section 1, by means of the frequency-domain approach formulated by Moiola and Chen [8], the existence of Hopf bifurcation parameter is determined and shown that Hopf bifurcation occurs when the bifurcation parameter exceeds a critical value. The length of delay preserving the stability of the positive equilibrium is estimated in Section 2. In Section 3, some numerical simulation are carried out to verify the correctness of theoretical analysis results. Finally, some conclusions and discussions are given in Section 4.

1 Existence of Hopf bifurcation

In Eq. (3), let

$$\begin{cases} x_1(t) = \int_{-\infty}^{t} (t-s)\mathrm{e}^{-(t-s)} N(s)\mathrm{d}s \\ x_2(t) = \int_{-\infty}^{t} \mathrm{e}^{-(t-s)} N(s)\mathrm{d}s \end{cases} \tag{4}$$

Then (3) becomes

$$\begin{cases} \frac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)[\alpha - \beta N(t-\tau) - \gamma x_1(t)] \\ \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -x_1(t) + x_2(t) \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = N(t) - x_2(t) \end{cases} \tag{5}$$

It is easy to see that Eq. (5) has a unique positive equilibrium $E_0(x_0, x_0, x_0)$, where $x_0 = \frac{\alpha}{\beta+\gamma}$.

We can rewrite the nonlinear system (5) in a matrix form as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + H(x) \tag{6}$$

where $x = (N(t), x_1(t), x_2(t))^{\mathrm{T}}$

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$H(x) = \begin{pmatrix} -\beta N(t)N(t-\tau) - \gamma N((t)x_1(t) \\ 0 \\ 0 \end{pmatrix}$$

Choosing the coefficient τ as a bifurcation and introducing a "state-feedback control" $u = g(y(t-\tau); \tau)$, where $y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$, we obtain a linear system with a non-linear feedback as follows

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bu \\ y = -C_x \\ u = g(y(t-\tau); \tau) \end{cases} \tag{7}$$

where

$$B = C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, u[g(y-\tau), \tau] = \begin{pmatrix} -\beta y_1(t)y_1(t-\tau) - \gamma y_1(t)y_2(t) \\ 0 \\ 0 \end{pmatrix}$$

Next, taking Laplace transform on (7), we obtain the standard transfer matrix of the linear part of the system:

$$G(s; \tau) = C[sI - A]^{-1} B$$

Then

$$G(s; \tau) = \begin{pmatrix} \frac{1}{s-\alpha} & 0 & 0 \\ \frac{1}{(s-\alpha)(s+1)^2} & \frac{1}{s+1} & \frac{1}{(s+1)^2} \\ \frac{1}{(s+1)(s-\alpha)} & 0 & \frac{1}{(s+1)} \end{pmatrix} \tag{8}$$

If this feedback system is linearized about the equilibrium $y = \tilde{y} = -C(x_0, x_0, x_0)^{\mathrm{T}}$, then the Jacobian of (8) is given by

$$J(\tau) = \frac{\partial g}{\partial y}\bigg|_{y=\tilde{y}=-C(x_0, x_0, x_0)^{\mathrm{T}}} =$$

$$\begin{pmatrix} 2x_0\beta e^{-s\tau}+\gamma x_0 & \gamma x_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (9)$$

Set

$$h(\lambda, s; \tau) = \det|\lambda I - G(s;\tau)J(\tau)| = \lambda^2 - \left[\frac{x_0\beta e^{-s\tau}+\gamma x_0}{s-\alpha} + \frac{\gamma x_0}{(s+1)^2(s-\alpha)}\right]\lambda = 0 \quad (10)$$

Then, we obtain the following results by applying the generalized Nyquist stability criterion with $s = i\omega$.

Lemma[8] If an eigenvalue of the corresponding Jacobian of the nonlinear system, in the time domain, assumes a purely imaginary value $i\omega_0$ at a particular $\tau=\tau_0$, then the corresponding eigenvalue of the constant matrix $G(i\omega_0;\tau_0)J(\tau_0)$ in the frequency domain must assume the value $-1+i0$ at $\tau=\tau_0$.

To apply Lemma 1, let $\lambda=\lambda(i\omega;\tau)$ be the eigenvalue of $G(i\omega;\tau)J(\tau)$ that satisfies $\lambda(i\omega_0;\tau_0) = -1+0i$. Then

$$h(-1, i\omega_0; \tau_0) = 1 + \frac{x_0\beta e^{-i\omega_0\tau_0}+\gamma x_0}{i\omega_0-\alpha} + \frac{\gamma x_0}{(i\omega_0+1)^2(i\omega_0-\alpha)} = 0 \quad (11)$$

Separating the real and imaginary parts, we obtain

$$(1-\omega_0^2)x_0\beta\cos\omega_0\tau_0 + 2\omega_0 x_0\beta\sin\omega_0\tau_0 = \alpha - \omega_0^2\alpha + 2\omega_0^2 - 2\gamma x_0 + \omega_0^2\gamma x_0 \quad (12)$$

$$2_{\omega_0} x_0\beta\cos\omega_0\tau_0 - (1-\omega_0^2)x_0\beta\sin\omega_0\tau_0 = 2\omega_0\alpha - \omega_0 + \omega_0^3 - 2\omega_0\gamma x_0 \quad (13)$$

Then we have

$$P_1^2+P_2^2 = P_3^2+P_4^2 \quad (14)$$

where

$$P_1=(1-\omega_0^2)x_0\beta,\ P_2 = 2\omega_0 x_0\beta,$$
$$P_3 = \alpha - \omega_0^2\alpha + 2\omega_0^2 - 2\gamma x_0 + \omega_0^2\gamma x_0,\ P_4 = 2\omega_0\alpha - \omega_0 + \omega_0^3 - 2\omega_0\gamma x_0$$

From (14), we can calculate the value of ω_0, then from (12) and (13), we can obtain

$$\tau k = \frac{1}{\omega_0}\left[2k\pi + \arcsin\frac{P_3}{\sqrt{P_1^2+P_2^2}} - \theta\right](k=0,1,2,\cdots) \quad (15)$$

where θ satisfies

$$\tan\theta = \frac{1-\omega_0^2}{2\omega_0} \quad (16)$$

According to the discussion above, we have the following conclusion.

Theorem 1 (**Existence of Hopf bifurcation parameter**) For system (5), If ω_0 is the positive real root of (14), then Hopf bifurcation point is

$$\tau_k = \frac{1}{\omega_0}\left[2k\pi + \arcsin\frac{P_3}{\sqrt{P_1^2+P_2^2}} - \theta\right](k=0,1,2,\cdots)$$

where θ satisfies (16).

2 Estimation of the length of delay to preserve stability

In the present section, we will obtain an estimation τ_+ for the length of the delay τ which preserves the stability of the positive equilibrium $E_0(x_0, x_0, x_0)$, i. e., $E_0(x_0, x_0, x_0)$ is asymptotically stable if $\tau < \tau_+$. In order to obtain our result, we assume that

(H) $x_0\beta < 2$

It is easy to obtain that the linearization of Eq. (5) near $E_0(x_0, x_0, x_0)$

$$\begin{cases} \frac{dN(t)}{dt} = -x_0\beta N(t-\tau) - \gamma x_0 x_1(t) \\ \frac{dx_1(t)}{dt} = -x_1(t) + x_2(t) \\ \frac{dx_2(t)}{dt} = N(t) - x_2(t) \end{cases} \quad (17)$$

We consider system (5) in $C([-\tau, \infty), R^3)$ with the initial values

$$N(\xi) = \varphi_1(\xi),\ x_1(\xi) = \varphi_2(\xi),\ x_2(\xi) = \varphi_3(\xi),\ \varphi_i(0) \geqslant 0,\ i=1,2,3,\ \xi\in[-\tau, 0]$$

Taking Laplace transform of system (17), we get

$$\begin{cases} s\widetilde{N} = -x_0\beta e^{-s\tau}M(s) - x_0\beta e^{-s\tau}\widetilde{N} - \gamma x_0\tilde{x}_1 + \varphi_1(0) \\ (s+1)\tilde{x}_1 = \tilde{x}_2 + \varphi_2(0) \\ (s+1)\tilde{x}_2 = \widetilde{N} + \varphi_3(0) \end{cases} \quad (18)$$

where $\widetilde{N}$, $\tilde{x}_1$ and $\tilde{x}_2$ are the Laplace transform of $\widetilde{N}(t)$, $x_1(t)$ and $x_2(t)$, respectively, and $M(s) = \int_{-\tau}^{0} e^{-st}N(t)dt$. Solving (18) for $\widetilde{N}$ leads to $\widetilde{N}_1 = \frac{K(s,\tau)}{J(s)}$, where

$$K(s,\tau) = (s+1)^2[x_0\beta e^{-s\tau}M(s) + \varphi_1(0)] + \gamma x_0\varphi_3(0) - \gamma x_0(s+1)\varphi_2(0)$$

$$J(s) = (s+x_0\beta e^{-s\tau})(s+1)^2 + \gamma x_0$$

Following along the lines of [11] and using the Nyquist criterion, we obtain that the conditions for local asymptotic stability of $E_0(x_0, x_0, x_0)$ are given by

$$\mathrm{Im}\{J(\mathrm{i}\omega_0)\} > 0 \quad (19)$$

$$\mathrm{Re}\{J(\mathrm{i}\omega_0)\} = 0 \quad (20)$$

where $\mathrm{Im}\{J(\mathrm{i}\omega_0)\}$ and $\mathrm{Re}\{J(\mathrm{i}\omega_0)\}$ are the imaginary part and real part of $J(\mathrm{i}\omega 0)$, respectively and ω_0 is the small positive root of (20). It follows from (19) and (20) that

$$\omega_0 - \omega_0^3 > (1-\omega_0^2)x_0\beta\sin\omega_0\tau - 2\omega_0 x_0\beta\cos\omega_0\tau \quad (21)$$

$$2\omega_0^2 - \gamma x_0 = (1-\omega_0^2)x_0\beta\cos\omega_0\tau - 2\omega_0 x_0\beta\sin\omega_0\tau \quad (22)$$

From (22), we obtain

$$2\omega_0^2 - \gamma x_0 \leqslant (1+\omega_0^2)x_0\beta + 2\omega_0 x_0\beta \quad (23)$$

Then

$$(2 - x_0\beta)\omega_0^2 - 2x_0\beta\omega_0 - (\beta+\gamma)x_0 \leqslant 0 \quad (24)$$

which leads to $\omega_0 \leqslant \omega_+$, where

$$\omega_+ = \frac{2x_0\beta + \sqrt{(2x_0\beta)^2 + 4(2-x_0\beta)(\beta+\gamma)x_0}}{2(2-x_0\beta)}.$$

By (21), we have

$$\omega_0^2 < 1 - \frac{(1-\omega_0^2)x_0\beta}{\omega_0}\sin\omega_0\tau + 2x_0\beta\cos\omega_0\tau$$

Hence

$$2\omega_0^2 < 2 - \frac{2(1-\omega_0^2)x_0\beta}{\omega_0}\sin\omega_0\tau + 4x_0\beta\cos\omega_0\tau \quad (25)$$

In view of (22), we get

$$2\omega_0^2 = (1-\omega_0^2)x_0\beta\cos\omega_0\tau - 2\omega_0 x_0\beta\sin\omega_0\tau + \gamma x_0 \quad (26)$$

Substituting (26) into (25) and rearranging, we get

$$(-\omega_0^2 - 3)x_0\beta\cos\omega_0\tau + \left[\frac{2(1-\omega_0^2)x_0\beta}{\omega_0} - 2x_0\beta\right]\cdot \sin\omega_0\tau < 2 - \gamma x_0 \quad (27)$$

It follows from (27) that

$$(-\omega_0^2 - 3)x_0\beta(\cos\omega_0\tau - 1) + \left[\frac{2(1-\omega_0^2)x_0\beta}{\omega_0} - 2x_0\beta\right]\cdot \sin\omega_0\tau < 5 + \omega_+^2 - \gamma x_0 \quad (28)$$

Using the bounds

$$(-\omega_0^2 - 3)x_0\beta(\cos\omega_0\tau - 1) = 2(\omega_0^2+3)\sin^2\left(\frac{\omega_0\tau}{2}\right) \leqslant \frac{1}{2}(\omega_+^2+3)\omega_+^2 + \tau^2$$

and

$$\left[\frac{2(1-\omega_0^2)x_0\beta}{\omega_0} - 2x_0\beta\right]\sin\omega_0\tau \leqslant (2\omega_+^2 x_0\beta + 4x_0\beta)\tau$$

we obtain

$$L_1\tau^2 + L_2\tau \leqslant L_3$$

where

$$L_1 = \frac{1}{2}(\omega_+^2+3)\omega_+^2,\ L_2 = (2\omega_+^2 + x_0\beta + 4x_0\beta),\ L_3 = 5 + \omega_+^2 - \gamma x_0$$

It is easy to see that if $\tau < \tau_+ = \dfrac{-L_2 + \sqrt{L_2^2 + 4L_1L_3}}{2L_1}$, the stability of $E_0(x_0, x_0, x_0)$ of system (5) is preserved. Thus we are now in a position to state the following result.

Theorem 2 Suppose that (H) holds. If there exists a parameter $0 \leqslant \tau < \tau_+$ such that $L_1\tau^2 + L_2\tau \leqslant L_3$, then τ_+ is the maximum value (length of delay) of τ for which $E_0(x_0, x_0, x_0)$ of system (5) is asymptotically stable.

3 Numerical examples

In this section, we present some numerical results of system (5) to verify the analytical predictions obtained in the previous section. Let us consider the following system:

$$\begin{cases}\dfrac{\mathrm{d}N(t)}{\mathrm{d}t} = N(t)[1 - 0.5N(t-\tau) - 0.3x_1(t)] \\ \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -x_1(t) + x_2(t) \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = N(t) - x_2(t)\end{cases} \quad (29)$$

which has a positive equilibrium $E_0(x_0, x_0, x_0) = (\frac{5}{4}, \frac{5}{4}, \frac{5}{4})$ and satisfies the conditions indicated in Theorem 1. Take $k = 0$ for example, by some computation by means of Matlab 7.0, we get $\tau_0 \approx 1.95$. Thus, the positive equilibrium $E_0 = (\frac{5}{4}, \frac{5}{4}, \frac{5}{4})$ is stable when $\tau < \tau_0$ which is illustrated by the computer simulations (see Figs. 1～7). When τ passes through the critical value τ_0, the positive equilibrium $E_0 = (\frac{5}{4}, \frac{5}{4}, \frac{5}{4})$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium $E_0 = (\frac{5}{4}, \frac{5}{4}, \frac{5}{4})$, which are depicted in Figs. 8～14.

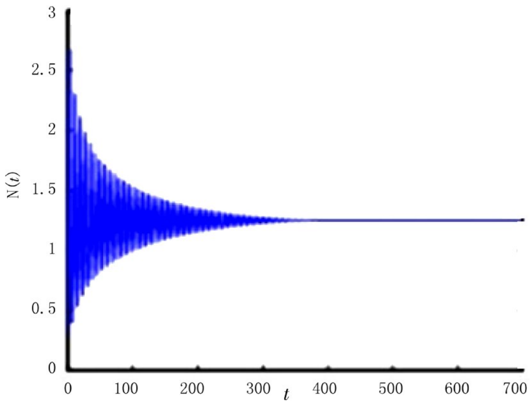

Fig. 1 **Dynanicial behavior of system**(4.1)

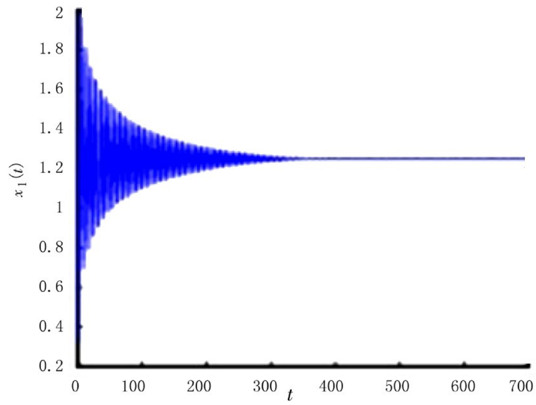

Fig. 2 **Dynanicial behavior of system**(4.1)

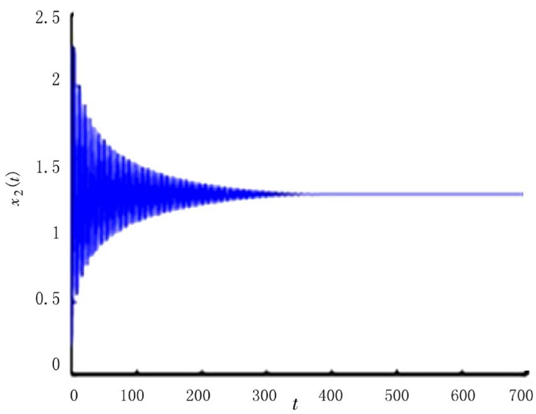

Fig. 3 **Dynanicial behavior of system**(4.1)

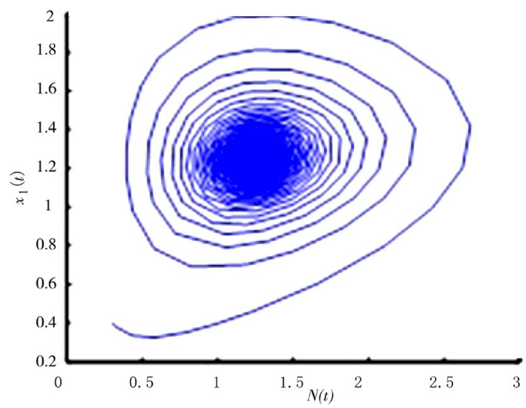

Fig. 4 **Dynanicial behavior of system**(4.1)

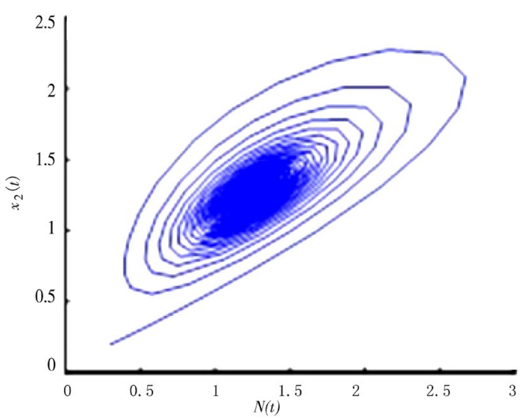

Fig. 5 **Dynanicial behavior of system**(4.1))

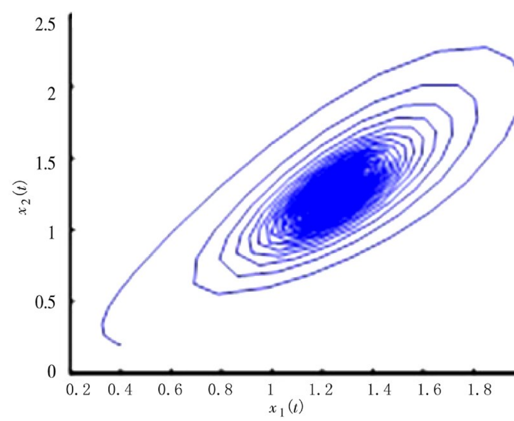

Fig. 6 **Dynanicial behavior of system**(4.1)

Figs. 1-7 Behavior and phase portrait of system (29) with $\tau = 1.9 < \tau_0 \approx 1.95$. The positive equilibrium $E_0 = (\frac{5}{4}, \frac{5}{4}, \frac{5}{4})$ is asymptotically stable. The initial value is (0.3, 0.4, 0.2).

Figs. 8～14 Behavior and phase portrait of system (29) with $\tau = 2 > \tau_0 \approx 1.95$. Hopf bifurcation occurs from the positive equilibrium $E_0 = (\frac{5}{4}, \frac{5}{4}, \frac{5}{4})$. The initial value is (0.3, 0.4, 0.2).

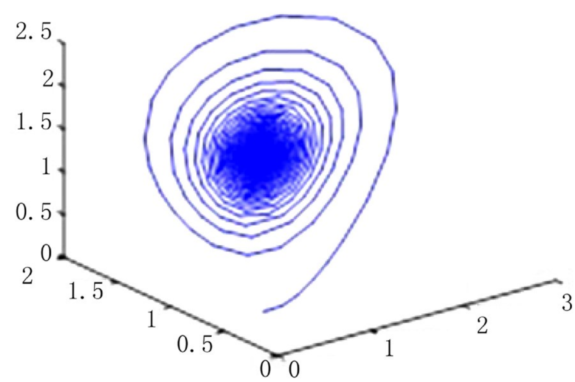

Fig. 7 **Dynanicial behavior of system**(4.1)

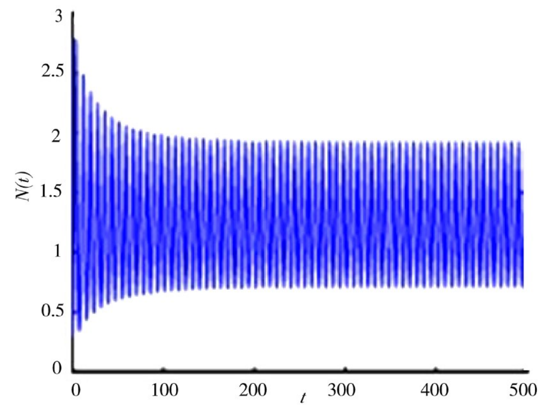

Fig. 8　**Dynanicial behavior of system**(4.1)

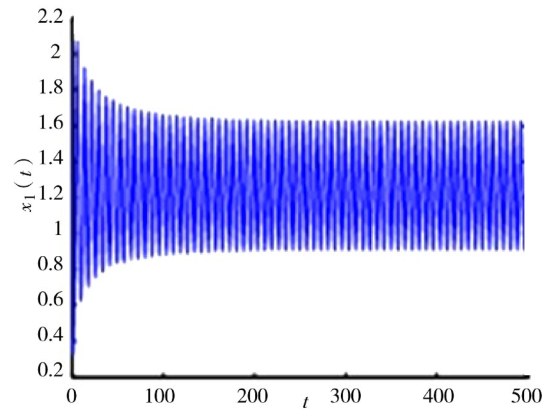

Fig. 9　**Dynanicial behavior of system**(4.1)

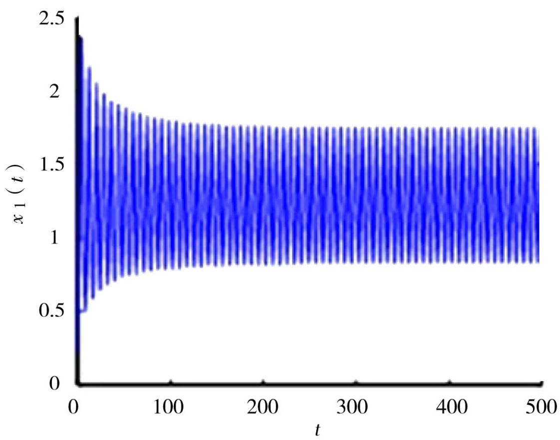

Fig. 10　**Dynanicial behavior of system**(4.1)

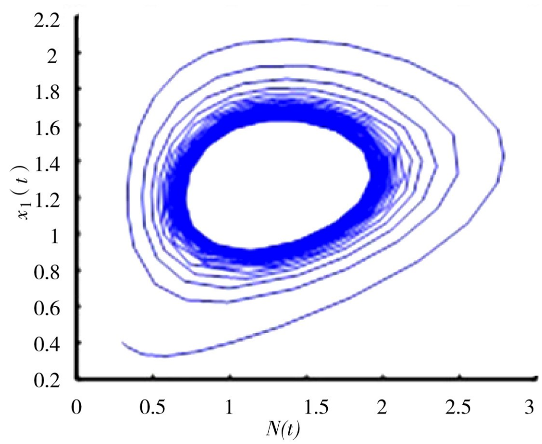

Fig. 11　**Dynanicial behavior of system**(4.1)

4 Conclusions and discussions

In this paper, we investigated a class of Logistic model with delay. By choosing the coefficient τ as a bifurcating parameter and analyzing the associating characteristic equation. It is found that a Hopf bifurcation occurs when the bifurcating parameter τ passes through a critical value. Meanwhile, the length of delay preserving the stability of the positive equilibrium $E_0(x_0, x_0, x_0)$ is estimated. Considering computational complexity, the direction and the stability of the bifurcating periodic orbits of system (3) have not been studied. It is beyond the scope of the present paper and will be further investigated elsewhere in the future.

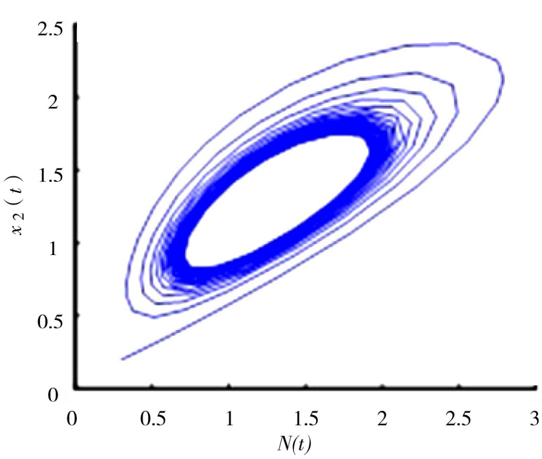

Fig. 12　**Dynanicial behavior of system**(4.1)

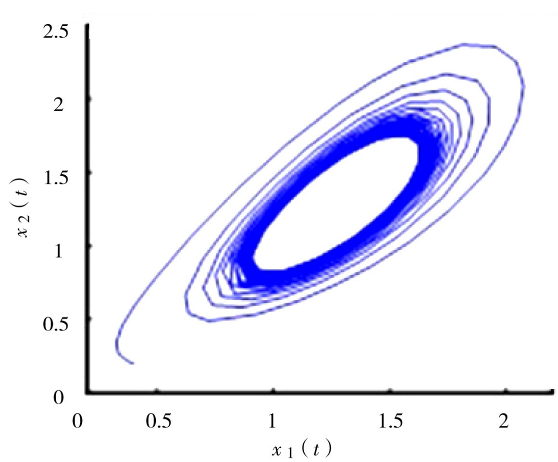

Fig. 13　**Dynanicial behavior of system**(4.1)

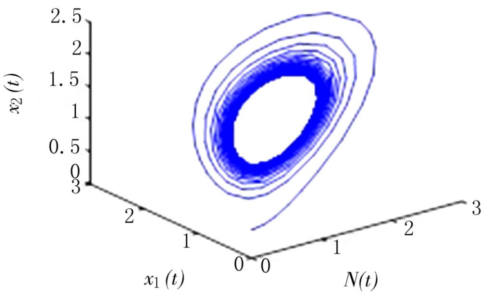

Fig. 14 **Dynanicial behavior of system**(4.1)

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具有时滞的 Logistic 模型的分支问题的频域分析

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摘要:研究一类具有时滞的 Logistic 模型。运用频域法并分析该模型对应的特征方程,得到了系统 Hopf 分支点,通过选择时滞 τ 作为参数,发现当参数通过某一临界值时,Hopf 分支产生,Hopf 分支产生,同时得到了系统正平衡点稳定的时滞范围处于零与某个正常数之间,数值模拟验证当时滞 $\tau=\tau_0$ 时,系统的正平衡点是局部稳定的,当 $\tau>\tau_0$ 时,系统的正平衡点是不稳定的。

关键词: Logistic 模型; 稳定性; Hopf 分支; 频域; 时滞

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