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Stability and Hopf Bifurcation Analysis in a Mutualistic System with Delay*

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Abstract: In this paper, a two-species Lotka-Volterra mutualistic system with delay $\begin{cases}\dot{x}(t)=r_1x(t)[1-a_1x(t-\tau)+a_2y(t)]\\ \dot{y}(t)=r_2y(t)[1+a_3x(t)-a_4y(t)]\end{cases}$ is considered. The existence of Hopf bifurcations at the positive equilibrium(x_*, y_*) when $\tau=\tilde{\tau}$, is established by analyzing the distribution of the characteristic values. An explicit algorithm for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions is derived, by using the normal form theory and center manifold argument. Finally, some numerical simulations are carried out for supporting the analytic results.

Key words: mutualistic system; delay; Hopf bifurcation; periodic solutions

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1 Introduction

There have been several previous reports dealing with stability and Hopf bifurcation in a competitive Lotka-Volterra system with delays[1-4]. In 2004, Song et al.[3] considered the delayed competition system. Taking the delay as the bifurcation parameter, they analyzed the stability of the interior positive equilibrium and the existence of local Hopf bifurcation. In 2009, Zhang et al.[5] considered a maturation delay in the same system, their results show that Hopf bifurcation arisen from the positive equilibrium as delay increase and crosses a critical value.

We are interesting the mutualistic system with delay. The two-species Lotka-Volterra mutualistic system with delay be modeled by the following system

$$\begin{cases}\dot{x}(t)=r_1x(t)[1-a_1x(t-\tau)+a_2y(t)]\\ \dot{y}(t)=r_2y(t)[1+a_3x(t)-a_4y(t)]\end{cases} \tag{1}$$

where $r_j(j=1,2)$, $a_i(i=1,2,3,4)$, τ are positive constants, $x(t)$ or $y(t)$ can be interpreted as the densities of certain species, τ is maturation time. For convenience letting $\tilde{x}=a_1x$, $\tilde{y}=a_4y$, and dropping the tildes for simplification of notation, (1) is transformed into

$$\begin{cases}\dot{x}(t)=r_1x(t)[1-x(t-\tau)+ay(t)]\\ \dot{y}(t)=r_2y(t)[1+bx(t)-y(t)]\end{cases} \tag{2}$$

where $a=\dfrac{a_2}{a_4}>0$, $b=\dfrac{a_3}{a_1}>0$. To be specific and for simplicity, we consider only system (2). Due to the biological interpretation of system (2), we always suppose that under the condition (H) $0<a<1$, $0<b<1$, system (2) has a positive equilibrium E_* denoted by

$$x_*=\frac{1+b}{1-ab}, y_*=\frac{1+a}{1-ab}$$

In the present paper, we investigate the effect of delay on dynamics of Eq. (2). Taking the delay τ as a parameter, we show that if the positive equilibrium E_* of system (2) withoutdelay is asymptotically stable, then the positive equilibrium E_* of system (2) always stable for all $\tau\geqslant0$. However, when the positive equilibrium E_* of system (2) without delay is unstable, then Hopf bifurcation occur as delay τ pas-

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ses through a critical value, i.e., a family of periodic solutions will be bifurcated from the positive equilibrium.

The rest of this paper is organized as follows. In section 2, we consider the stability and local Hopf bifurcation of the positive equilibrium. In section 3, we employ the normal form method and the center manifold theory introduced by Hassard et al.[6] to analyze the direction, stability and the period of the bifurcating periodic solution at the critical values of τ.

2 The Existence of Hopf Bifurcation

By the translation $u_1(t)=x(t)-x_*$, $u_2(t)=y(t)-y_*$, system (2) is transformed into

$$\begin{cases}\dot{u}_1(t)=r_1(u_1(t)+x_*)[u_1(t-\tau)-au_2(t)]\\ \dot{u}_2(t)=r_2(u_2(t)+y_*)[-bu_1(t)+u_2(t)]\end{cases} \tag{3}$$

The linearization of (4) at $u=0$ is

$$\begin{cases}\dot{u}_1(t)=-r_1x_*u_1(t-\tau)+ar_1x_*u_2(t)\\ \dot{u}_2(t)=-r_2y_*u_2(t)+br_2y_*u_1(t)\end{cases} \tag{4}$$

Thus, the characteristic equation of (5) is

$$\lambda^2+p\lambda-r+(s\lambda+q)\mathrm{e}^{-\lambda\tau}=0 \tag{5}$$

where $p=r_1x_*$, $s=r_2y_*$, $q=r_1r_2x_*y_*$, $r=abr_1r_2x_*y_*$. The stability of the trivial solution of system (4) depends on the locations of roots of characteristic Eq. (6) when all roots of Eq. (5) locate in the left half of the complex plane, the trivial solution of system (4) is stable; otherwise, it is unstable. In the following, we will investigate the distribution of roots of Eq. (5). Noting that $\lambda=0$ is not a solution to (5), and when $\tau=0$, Eq. (5) becomes

$$\lambda^2+(p+s)\lambda+q-r=0 \tag{6}$$

we have the following result.

Lemma 1 When $\tau=0$, if the condition (H) holds, Eq.(6) has two negative real roots.

In addition, we know that $\mathrm{i}\omega(\omega>0)$ is a root of Eq. (5) if and only if ω satisfies the following equation

$$-\omega^2-r+\mathrm{i}p\omega+(\mathrm{i}s\omega+q)(\cos\omega\tau-\mathrm{i}\sin\omega\tau)=0$$

Separating the real and imaginary parts, we have

$$\begin{cases}\omega^2+r=s\omega\sin\omega\tau+q\cos\omega\tau\\ p\omega=q\sin\omega\tau-s\omega\cos\omega\tau\end{cases} \tag{7}$$

This leads to

$$\omega^4+(2r-s^2+p^2)\omega^2+r^2-q^2=0 \tag{8}$$

When the condition (H) holds, it is easy to see that $r^2-q^2<0$ and $\Delta=(2r-s^2+p^2)-4(r^2-q^2)>0$, then Eq. (8) has only one positive root $\omega_0=\sqrt{\frac{s^2-p^2-2r+\sqrt{\Delta}}{2}}$. From (7), we can determine

$$\tau_j=\frac{1}{\omega_0}\cos^{-1}\frac{(q-ps)\omega_0^2+qr}{q^2+s^2\omega_0^2}+\frac{2j\pi}{\omega_0},j=0,1,2,\cdots \tag{9}$$

then (5) with $\tau=\tau_j$ has a pair of purely imaginary roots $\pm\mathrm{i}\omega_0$, and $\tau_0<\tau_1<\tau_2<\cdots$.

Summarizing the above discussion, we have the following result.

Lemma 2 When the condition (H) holds

1) if $\tau\in(0,\tau_0)$, then all roots of Eq. (6) have strictly negative real parts;

2) if $\tau=\tau_j$, then all roots of Eq. (6) has a simple pair of purely imaginary roots $\pm\mathrm{i}\omega_j$, $j=0,1,2,\cdots$.

Let $\lambda(\tau)=\alpha(\tau)+\mathrm{i}\omega(\tau)$ be the root of Eq. (6) satisfying $\alpha(\tau_j)=0$, $\omega(\tau_j)=\omega_0$, $j=0,1,2,\cdots$; then we have the following result.

Lemma 3 When the condition (H) holds, the following transversality conditions are satisfied

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Big|_{\tau=\tau_j}>0,j=0,1,2,\cdots$$

Proof Differentiating the two sides of Eq. (5) with respect to τ, we have

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}=\frac{(2\lambda+p)\mathrm{e}^{\lambda\tau}+s}{\lambda(s\lambda+q)}-\frac{\tau}{\lambda}$$

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Big|_{\tau=\tau_j}=\frac{\sqrt{\Delta}}{s^2\omega_0^2+q^2}>0(j=0,1,2,\cdots)$$

The proof is complete.

Lemma 4 When the condition (H) holds, if $\tau>\tau_0$, then Eq. (H) has at least one root with strictly positive real part.

Applying Lemmas 1～4, we can get the following the stability and bifurcation results for system (2).

Theorem 1 For system (2), when the condition (H) holds,

1) if $\tau\in(0,\tau_0)$, then the positive equilibrium E_* is local asymptotically stable;

2) if $\tau>\tau_0$, then he positive equilibrium E_* is unstable and system (2) undergoes a Hopf bifurcation at E_* when $\tau=\tau_j$, $j=0,1,2,\cdots$.

3 The Direction and Stability of the Hopf Bifurcation

In the previous sections, we obtained conditions for Hopf bifurcation to occur when $\tau=\tau_j$, $j=0,1,2,\cdots$. In this section, we shall study the direction and stability of the Hopf bifurcation by using techniques from normal form and center manifold theory[6]. For convenience, we denote the critical value of τ, derived by (9), as $\tilde{\tau}$ and let $\tau=\tilde{\tau}+\mu$, $\mu\in\mathbf{R}$. Then $\mu=0$ is the Hopf bifurcation value of the system (2).

Let $u_1(t)=x(\tau t)-x_*$, $u_2(t)=y(\tau t)-y_*$; then system (2) may be written as a FED in $C=C([-1,0],\mathbf{R}^2)$ as

$$\begin{cases}\dot{u}_1(t)=\tau r_1(u_1(t)+x_*)[-u_1(t-1)+au_2(t)]\\ \dot{u}_2(t)=\tau r_2(u_2(t)+y_*)[bu_1(t)-u_2(t)]\end{cases}\tag{10}$$

For $\phi=(\phi_1,\phi_2)^{\mathrm{T}}\in C$, let

$$L_\mu\phi=(\tilde{\tau}+\mu)\begin{pmatrix}0 & r_1ax_*\\ r_2by_* & -r_2y_*\end{pmatrix}\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\end{pmatrix}+(\tilde{\tau}+\mu)\begin{pmatrix}-r_1x_* & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\end{pmatrix}$$

and

$$f(\mu,\phi)=(\tilde{\tau}+\mu)\begin{pmatrix}-r_1\phi_1(0)\phi_1(-1)+ar_1\phi_1(0)\phi_2(0)\\ br_2\phi_1(0)\phi_2(0)-r_2\phi_2^2(0)\end{pmatrix}$$

By the Riesz representation theorem, there exists a 2×2 matrix, $\eta(\theta,\mu)(-1\leqslant\theta\leqslant0)$, whose elements are of bounded variation functions such that

$$L_\mu(\phi)=\int_{-1}^{0}[\mathrm{d}\eta(\theta,\mu)]\phi(\theta)$$

In fact, we can choose

$$\eta(\theta,\mu)=(\tilde{\tau}+\mu)\begin{pmatrix}0 & r_1ax_*\\ r_2by_* & -r_2y_*\end{pmatrix}\delta(\theta)-(\tilde{\tau}+\mu)\begin{pmatrix}-r_1x_* & 0\\ 0 & 0\end{pmatrix}\delta(\theta+1)$$

For $\phi\in C^1([-1,0],\mathbf{R}^2)$, define the operator $A(\mu)$ as

$$A(\mu)\phi(\theta)=\begin{cases}\dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, \theta\in[-1,0)\\ \displaystyle\int_{-1}^{0}[\mathrm{d}\eta(\xi,\mu)]\phi(\xi), \theta=0\end{cases}$$

$$R(\mu)\phi(\theta)=\begin{cases}0, \theta\in[-1,0)\\ h(\mu,\phi), \theta=0\end{cases}$$

Then system (2) is equivalent to the following operator equation

$$\dot{u}_t=A(\mu)x_t+R(\mu)x_t$$

For $\psi\in C^1([0,1],(\mathbf{R}^2)^*)$, define

$$A^*\psi(s)=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, s\in(0,1]\\ \displaystyle\int_{-1}^{0}\psi(-\xi)\mathrm{d}\eta(\xi,0), s=0\end{cases}$$

and a bilinear form

$$\langle\psi(s),\phi(\theta)\rangle=\bar{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi$$

where $\eta(\theta)=\eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. From the discussion in Section 2, we know that $\pm\mathrm{i}\omega_0\tilde{\tau}$ are eigenvalues of $A(0)$ and therefore they are also eigenvalues of A^*. It is not difficult to verify that the vector $q(\theta)=(1,\alpha)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}\theta}(\theta\in[-1,0])$ and $q^*(s)=D(\beta,1)\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}\theta}(s\in[0,1])$ are the eigenvectors $A(0)$ and A^* corresponding to the eigenvalue $\mathrm{i}\omega_0\tilde{\tau}$ and $-\mathrm{i}\omega_0\tilde{\tau}$, respectively, where

$$\alpha=\frac{\mathrm{i}\omega_0\tilde{\tau}+r_1x_*\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}}{r_1ax_*},\beta=\frac{-\mathrm{i}\omega_0\tilde{\tau}+r_2y_*}{r_1ax_*}$$

$$D=\left(\frac{-2\mathrm{i}\omega_0\tilde{\tau}+r_2y_*+(1-\omega_0\tilde{\tau}^2-\tilde{\tau}r_2y_*)r_1x_*\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}}{r_1ax_*}\right)^{-1}$$

Following the algorithms given in Hassard et al.[6] and using a computation process similar to that of Wei and Li[7], we can obtain the coefficients which will be used in determining the important quantities:

$$\begin{aligned}g_{20}&=2\bar{D}\tilde{\tau}(-r_1\bar{\beta}\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}+r_1a\alpha\bar{\beta}+r_2b\alpha-r_2\alpha^2)\\ g_{11}&=\bar{D}\tilde{\tau}(-r_1\bar{\beta}\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}-r_1\bar{\beta}\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}+2r_1a\alpha\bar{\beta}+2r_2b\alpha-2r_2\alpha^2)\\ g_{02}&=2\bar{D}\tilde{\tau}(-r_1\bar{\beta}\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}+r_1a\alpha\bar{\beta}+r_2b\alpha-r_2\alpha^2)\\ g_{21}&=2\bar{D}\tilde{\tau}\Big\{r_1\bar{\beta}\Big[-\frac{1}{2}W_{20}^{(1)}(0)\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}-W_{11}^{(1)}(0)\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}-\\ &W_{11}^{(1)}(-1)-\frac{1}{2}W_{20}^{(1)}(-1)+\frac{1}{2}a\alpha W_{20}^{(1)}(0)+a\alpha W_{11}^{(1)}(0)+\\ &\frac{1}{2}aW_{20}^{(2)}(0)+aW_{11}^{(2)}(0)\Big]+r_2\Big[\frac{1}{2}b\alpha W_{20}^{(1)}(0)+b\alpha W_{11}^{(1)}(0)+\\ &\frac{1}{2}bW_{20}^{(2)}(0)+bW_{11}^{(2)}(0)-\alpha W_{20}^{(2)}(0)-2\alpha W_{11}^{(2)}(0)\Big]\Big\}\end{aligned}\tag{11}$$

where

$$W_{20}(\theta)=\frac{\mathrm{i}g_{20}}{\omega_0\tilde{\tau}}q(\theta)+\frac{\mathrm{i}\bar{g}_{02}}{3\omega_0\tilde{\tau}}\bar{q}(\theta)+E_1\mathrm{e}^{2\mathrm{i}\omega_0\tilde{\tau}\theta}$$

$$W_{11}(\theta)=-\frac{\mathrm{i}g_{11}}{\omega_0\tilde{\tau}}q(\theta)+\frac{\mathrm{i}\bar{g}_{11}}{\omega_0\tilde{\tau}}\bar{q}(\theta)+E_2$$

and

$$E_1=-2\tilde{\tau}\begin{pmatrix}2\mathrm{i}\omega_0\tilde{\tau}+r_1x_*\mathrm{e}^{-2\mathrm{i}\omega_0\tilde{\tau}} & -r_1ax_*\\ -r_2by_* & 2\mathrm{i}\omega_0\tilde{\tau}+r_2y_*\end{pmatrix}^{-1}\times\begin{pmatrix}r_1\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}-ar_1\alpha\\ -br_2\alpha+r_2\alpha^2\end{pmatrix}$$

$$E_2=\tilde{\tau}\begin{pmatrix}-r_1x_* & r_1ax_*\\ r_2by_* & -r_2y_*\end{pmatrix}^{-1}\times\begin{pmatrix}-r_1(\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}+\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}})+2ar_1\alpha\\ 2br_2\alpha-2r_2\alpha^2\end{pmatrix}$$

Consequently, g_{ij} in (11) can be expressed by the parameters and delay in (10). Thus, we can compute the following quantities

$$c_1(0)=\frac{\mathrm{i}}{2\omega_0\tilde{\tau}}\left(g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right)+\frac{g_{21}}{2}$$

$$\mu_2=-\frac{\mathrm{Re}(c_1(0))}{\mathrm{Re}(\lambda'(\tilde{\tau}))},\beta_2=2\mathrm{Re}(c_1(0))$$

$$T_2=-\frac{\mathrm{Im}(c_1(0))+\mu_2\mathrm{Im}(\lambda'(\tilde{\tau}))}{\omega_0\tilde{\tau}}$$

which determine the properties of bifurcating periodic solutions at the critical value $\tilde{\tau}$, i. e., μ_2 determines the directions of the Hopf bifurcation: if $\mu_2>0(\mu_2<0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau>\tilde{\tau}(\tau<\tilde{\tau})$; β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions on the center manifold are stable (unstable) if $\beta_2<0(\beta_2>0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2>0(T_2<0)$.

From the discussion in Section 2, we know that $\mathrm{Re}(\lambda'(\tilde{\tau}))>0$, therefore we have the following result.

Theorem 2 The direction of the Hopf bifurcation of the system(2) at the positive equilibrium (x_*, y_*) when $\tau=\tilde{\tau}$, with $\tilde{\tau}=\tau_j, j=1,2,\cdots$ is supercritical (subcritical) and the bifurcating periodic solutions are stable(unstable) if $\mathrm{Re}(c_1(0))<0(>0)$.

4 Numerical Simulations

In the previous section, we have derived an explicit algorithm for determining the direction of the Hopf bifurcations and stability of the bifurcating periodic solutions by using the center manifold theory and normal form method. In this section, we shall carry out some numerical simulations for supporting our theoretical analysis.

We choose $r_1=0.18, r_2=0.2, a=0.3, b=0.4$. From Section 2, we can obtain $(x_*, y_*)=(1.5909, 1.4773)$. We have carried out numerical simulations on system (2) using Matlab with these parameter values. Thus phase plots are shown in Fig. 1. The simulations support our theoretical result.

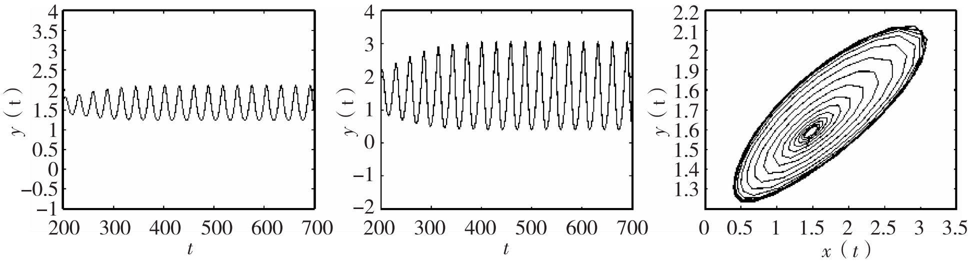

Fig. 1 Mathematical simulations of a periodic solution to system (2) with parameters specified by $r_1=0.18, r_2=0.2, a=0.3, b=0.4, \tau=6.6>\tau_0=4.8800$

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带有时滞的合作系统稳定性和 Hopf 分支分析

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摘要：本文研究了带有时滞的两个物种的合作系统 $\begin{cases}\dot{x}(t)=r_1x(t)[1-a_1x(t-\tau)+a_2y(t)]\\ \dot{y}(t)=r_2y(t)[1+a_3x(t)-a_4y(t)]\end{cases}$ 的稳定性和分支分析，通过分析特征根的分布得出系统在正平衡点(x_*, y_*)，当 $\tau=\tilde{\tau}$ 时存在 Hopf 分支，进一步应用规范型和中心流形的方法给出了计算分支周期解稳定性和方向的计算公式，最后通过数值模拟验证了理论结果的正确性。

关键词：合作系统；时滞；Hopf 分支；周期解

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