DOI:10.11721/cqnuj20130416

On Blow-up Solutions for a Modified Periodic Nonlinearly Dispersive Wave Equation*

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Abstract: In this paper, we investigate a generalized periodic nonlinearly dispersive wave equation by using the classical mathematical techniques. The local well-posedness for the equation is established by using the Kato's semigroup theory. Under suitable assumptions, on initial value, a precise blowup scenario (That is solution in a finite time if and only if $\lim_{t\uparrow T}\sup\{\sup_{x\in S}|\gamma u_x(t,x)|\}=+\infty$.) and a blow-up result to the equation are presented (That is a sufficient condition of blow-up).

Key words: a periodic nonlinear dispersive wave equation; local well-posedness; blowup scenario; blowup solution; the Camassa-Holm equation; the rod equation

中图分类号:O143 **文献标志码**:A **文章编号**:1672-6693(2013)04-0079-06

Introduction

Hu and Yin[1], and Yin[2] investigate the following equation

$$u_t-u_{txx}+2\tilde{\omega}u_x+3uu_x=\gamma(2u_xu_{xx}+uu_{xxx}) \tag{1}$$

where $\tilde{\omega}$ is nonnegative number and γ is arbitrary real number. It is shown in [1,2] that Eq. (1) has solitary wave solutions and blow-up solutions for nonperiodic case and also solutions which blow up in finite time for periodic case.

If $\gamma=0$, Eq. (1) becomes the famous BBM equation modelling the motion of internal gravity waves in shallow channel[3]. Some results related to the equation can be found in [4-5]. It is worthwhile to mention that the equation is not integrality and its solitary waves are not solitons[4].

If $\gamma=1$ in Eq. (1), the well-known Camassa-Holm equation, which models the unidirectional propagation of shallow water waves over a flat bottom, is found. Here, $u(t,x)$ represents the free surface above a flat bottom and $\tilde{\omega}$ is a nonnegative parameter related to the critical shallow water speed[6]. As a model to describe the shallow water motion, the Camassa-Holm equation posses a bi-Hamiltonian structure and in finite conservation laws[7-8] and is completely integrable[9]. Recently, some significant results of dynamical behaviors have been obtained for the Cauchy problem of the Camassa-Holm equation, the reader is referred to [10-12] and the references therein.

If $\tilde{\omega}=0$ and $\gamma\in\mathbf{R}$, Eq. (1) changes into the rod equation derived by Dai[13] recently, which describes finite-

* **Received**:10-18-2012 **Accepted**:11-15-2012
Foundation:Key Project of Natural Science of Education Sichuan Provincial(No. 10ZA008);RFSUSE(No. 2011KY12)
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收稿日期:2012-10-18 **修回日期**:2012-11-15 **网络出版时间**:2013-07-20 19:23
资助项目:四川省教育厅自然科学重点项目(No. 10ZA008);四川理工学院科研基金(No. 2011KY12)
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网络出版地址:http://www.cnki.net/kcms/detail/50.1165.N.20130720.1923.201304.79_013.html

length and small amplitude radial deformation waves in thin cylindrical compressible hyperelastic rods[13]. The first investigation of the Cauchy problem of the rod equation for nonperiodic case was done by Constantin and Strauss[14], the precise blow-up scenario, some blow-up results of strong solution and the stability of a class of solitary waves to the rod equation are presented. Yin[15] discusses the rod equation for periodic case and gives some interesting blow-up results.

We consider the Cauchy problem of the following periodic nonlinearly dispersive wave equation

$$\begin{cases} u_t-u_{txx}+2\bar{\omega}u_x+auu_x+\beta u_{xx}=\gamma(2u_xu_{xx}+uu_{xxx}),t>0,x\in\mathbf{R} \\ u(0,x)=u_0(x),x\in\mathbf{R} \\ u(t,x+1)=u(t,x),t>0,x\in\mathbf{R} \end{cases} \tag{2}$$

where $\bar{\omega}$, a and β are nonnegative fixed constants, γ is fixed arbitrary constant. Obviously, Eq. (2) reduces to Eq. (1) if we define $a=3$ and $\beta=0$. Actually, Wu and Yin[16] consider a nonlinearly dissipative Camassa-Holm equation which includes a nonlinearly dissipative term $L(u)$, where L is a differential operator. Thus, we can regard the term βu_{xx} as a dissipative term.

Because of the term βu_{xx}, Eq. (2) does not admit conservation laws in works [1-2], $E(u)=\int_S(u^2+u_x^2)\mathrm{d}x$. Several estimates are established to prove a blow-up solution. More precisely, we establish the local well-posedness of strong solutions for Eq. (2) subject to initial value $u_0\in H^r(S),r>\frac{3}{2}$ with $S=\frac{R}{Z}$(the circle of unit length) and give a precise blow-up scenario which is different from that in [1]. Under suitable assumptions on the initial value u_0, relying on the classical mathematical techniques, a sufficient condition about blow-up solutions is found. The results obtained in this paper improve considerably those in previous works[1, 17].

1 Local well-posedness

In this section, we establish the local well-posedness for the Cauchy problem (2) in $H^r(S),r>\frac{3}{2}$.

We denote by $*$ the convolution. Note that if $G(x):=\frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$, where $[x]$ stands for the integer part of $x\in\mathbf{R}$, then $(1-\partial_x^2)^{-1}f=G*f$ for all $f\in L^2(\mathbf{R})$ and $G*(u-u_{xx})=u$. Using this identity, equation (2) is written as

$$\begin{cases} u_t+\gamma uu_x+\partial_xG*\left[\frac{a-\gamma}{2}u^2+\frac{\gamma}{2}(u_x)^2+2\bar{\omega}u+\beta u_x=0,t>0,x\in\mathbf{R}\right. \\ u(0,x)=u_0(x) \\ u(t,x+1)=u(t,x) \end{cases} \tag{3}$$

which is equivalent to

$$\begin{cases} y_t+\gamma uy_x+2\gamma yu_x+2\bar{\omega}u_x+\beta u_{xx}+(a-3\gamma)uu_x=0,t>0,x\in\mathbf{R} \\ y(0,x)=u_0(x)-u_{0,xx}(x),x\in\mathbf{R} \\ y(t,x)=y(t,x+1),t>0,x\in\mathbf{R} \end{cases} \tag{4}$$

Theorem 1 Given $u_0\in H^r(S)(r>\frac{3}{2})$, there exists a maximal time $T=T(a,b,\gamma,\bar{\omega},u_0)$ and a unique solution u to problem (2), such that

$$u=u(\cdot,u_0)\in C([0,T);H^r(S))\cap C^1([0,T);H^{r-1}(S)) \tag{5}$$

Proof The proof of Theorem 1 can be finished by using the Kato's semigroup theory (see [1] or [2]). Here, we omit the detailed proof.

2 Blow-up solutions

Firstly, we give several useful lemmas.

Lemma 1　Let $s\geqslant\frac{3}{2}$ and $u(t,x)$ be the corresponding solution of equation (3) with initial data $u_0(x)\in H^r(S)$. It holds that if $q\in(0,r-1]$, there is a constant c depending only on q such that

$$\int_S(\Lambda^{q+1}u)^2\mathrm{d}x\leqslant\int_S(\Lambda^{q+1}u_0)^2\mathrm{d}x+c\int_0^t(\|\gamma u_x\|_{L^\infty}+1)(\|u\|_{H^q}^2+\|u\|_{H^{q+1}}^2)\mathrm{d}s \tag{6}$$

Proof　The proof of Lemma 1 can be finished by slightly modifying that of Lemma 4.6 in [18]. Here, we omit the detailed proof.

Lemma 2　Given $u_0\in H^r(S), r>\frac{3}{2}$, the solution $u(\cdot,u_0)$ of problem (2) blows up in finite time $T<+\infty$ if and only if $\lim_{t\uparrow T}\sup\{\sup_{x\in S}|\gamma u_x(t,x)|\}=+\infty$.

Proof　Applying (6) with $q=r-1$, we have

$$\|u\|_{H^r(S)}^2\leqslant\|u_0\|_{H^r(S)}^2+c\int_0^t(\|\gamma u_x\|_{L^\infty}+1)(\|u\|_{H^{r-1}(S)}^2+\|u\|_{H^r(S)}^2)\mathrm{d}s\leqslant\|u_0\|_{H^r(S)}^2+c\int_0^t(\|\gamma u_x\|_{L^\infty(S)}+1)\|u\|_{H^r(S)}^2\mathrm{d}s \tag{7}$$

It follows from (7) and the Gronwall's inequality that

$$\|u\|_{H^r(S)}^2\leqslant\|u_0\|_{H^r(S)}^2\exp\left(c\int_0^t(\|\gamma u_x\|_{L^\infty(S)}+1)\mathrm{d}s\right) \tag{8}$$

If there is a constant $M>0$ such that $\|\gamma u_x\|_{L^\infty}\leqslant M$ on $(0,T]$, then $\|u\|_{H^r(S)}^2$ does not blow up. It completes the proof of Theorem 4.

Remark 1　We use the technique differed from that in [1] to complete the proof of Lemma 2, it improves considerably the result obtained in [1].

Lemma 3　Let $u\in H^3(S)$ and $T>0$ be the maximal existence time of the solution $u(t,x)$ to problem (3). Then it holds (i) $\int_S u(t,x)\mathrm{d}x=\int_S u_0(x)\mathrm{d}x=\int_S y(t,x)\mathrm{d}x=\int_S y_0(x)\mathrm{d}x$; (ii) $\|u\|_{H^1(S)}^2\leqslant\|u_0\|_{H^1(S)}^2\mathrm{e}^{2\beta T}$.

Proof　The proof of (i) is similar to that of Lemma 6 in [2], so we omit it.

Multiplying u to both sides of equation (2) and integrating by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_S(u^2+u_x^2)\mathrm{d}x=\beta\int_S u_x^2\mathrm{d}x\leqslant\beta\int_S(u^2+u_x^2)\mathrm{d}x \tag{9}$$

which yields

$$\|u\|_{H^1(S)}^2\leqslant\|u_0\|_{H^1(S)}^2\mathrm{e}^{2\beta T} \tag{10}$$

It finishes the proof.

Lemma 4[17]　Assume that a differential function $y(t)$ satisfies

$$y'\leqslant-Cy^2(t)+K \tag{11}$$

with constants $C,K>0$. If the initial datum $y(0)=y_0<-\sqrt{\frac{K}{C}}$, then the solutions to (11) goes to $-\infty$ in finite time.

Next, we give the blow-up result.

Theorem 2　Assume that $u_0\in H^s(S), s>3$, and $\|u_0\|_{H^1}\neq0$.

(i) If $0<\gamma<\frac{a}{3}$ is such that

$$\int_S\gamma u_{0x}^3\mathrm{d}x<2\beta\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}-\sqrt{\frac{2}{3}}\left(\frac{\beta^2}{2}\|u_0\|_{H^1}^4\mathrm{e}^{4\beta T}+\frac{3\gamma(a-2\gamma)(\mathrm{e}+1)}{4(\mathrm{e}-1)}\times\|u_0\|_{H^1}^6\mathrm{e}^{6\beta T}+\left(12\tilde{\omega}\gamma+3\beta\gamma\frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\|u_0\|_{H^1}^5\mathrm{e}^{5\beta T}\right)^{\frac{1}{2}}$$

then the corresponding solution to equation (2) blows up infinite time.

(ii) If $\frac{a}{3}<\gamma<a$ is such that $\int_S \gamma u_{0x}^3 \mathrm{d}x < 2\beta \| u_0 \|_{H^1}^2 \mathrm{e}^{2\beta T} -$

$$\sqrt{\frac{2}{3}}\left(\frac{\beta^2}{2}\| u_0 \|_{H^1}^4 \mathrm{e}^{4\beta T} + \frac{3\gamma(a-\gamma)(\mathrm{e}+1)}{8(\mathrm{e}-1)}\| u_0 \|_{H^1}^6 \mathrm{e}^{6\beta T} + \left(12\tilde{\omega}\gamma + 3\beta\gamma \frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\| u_0 \|_{H^1}^5 \mathrm{e}^{5\beta T}\right)^{\frac{1}{2}}$$

then the corresponding solution to equation (2) blows up infinite time.

(iii) If $\gamma>a$ or $\gamma<0$ is such that $\int_S \gamma u_{0x}^3 \mathrm{d}x < 2\beta \| u_0 \|_{H^1}^2 \mathrm{e}^{2\beta T} -$

$$\sqrt{\frac{2}{3}}\left(\frac{\beta^2}{2}\| u_0 \|_{H^1}^4 \mathrm{e}^{4\beta T} + \frac{3\gamma(\gamma-a)(\mathrm{e}+1)}{8(\mathrm{e}-1)}\times \| u_0 \|_{H^1}^6 \mathrm{e}^{6\beta T} + \left(12\tilde{\omega}\gamma + 3\beta\gamma \frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\| u_0 \|_{H^1}^5 \mathrm{e}^{5\beta T}\right)^{\frac{1}{2}}$$

then the corresponding solution to equation (2) blows up infinite time.

Proof Let $T>0$ be the maximal time of existence of the solution u to equation (2) with the initial data u_0. Applying $\gamma u_x^2 \partial_x$ to both sides of equation (2) and integrating by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \gamma u_x^3 = -\frac{3\gamma^2}{2}\int_S u_x^4 \mathrm{d}x + 3\beta\int_S \gamma u_x^3 \mathrm{d}x + 3\,\frac{(a-\gamma)\gamma}{2}\int_S u_x^2 u^2 \mathrm{d}x + 6\tilde{\omega}\gamma\int_S u_x^2 u\,\mathrm{d}x -$$

$$3\int_S u_x^2 G * \left(\frac{(a-\gamma)\gamma}{2}u^2 + \frac{\gamma^2}{2}u_x^2 + 2\tilde{\omega}\gamma u + \beta\gamma u_x\right)\mathrm{d}x \tag{12}$$

Due to

$$\left|\int_S u_x^3 \mathrm{d}x\right| \leqslant \left(\int_S u_x^4 \mathrm{d}x\right)^{\frac{1}{2}}\left(\int_S u_x^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

thus

$$\int_S u_x^4 \mathrm{d}x \geqslant \frac{\left(\int_S u_x^3 \mathrm{d}x\right)^2}{\int_S u_x^2 \mathrm{d}x} \geqslant \frac{\left(\int_S u_x^3 \mathrm{d}x\right)^2}{\| u \|_{H^1}^2}$$

Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \gamma u_x^3 \leqslant -\frac{3}{2\| u_0 \|_{H^1}^2 \mathrm{e}^{2\beta T}}\left(\int_S \gamma u_x^3 \mathrm{d}x\right)^2 + 3\beta\int_S \gamma u_x^3 \mathrm{d}x + 3\,\frac{(a-\gamma)\gamma}{2}\int_S u_x^2 u^2 \mathrm{d}x + 6\tilde{\omega}\gamma\int_S u_x^2 u\,\mathrm{d}x -$$

$$3\int_S u_x^2 G * \left(\frac{(a-\gamma)\gamma}{2}u^2 + \frac{\gamma^2}{2}u_x^2 + 2\tilde{\omega}\gamma u + \beta\gamma u_x\right)\mathrm{d}x \tag{13}$$

Next, we divide into three cases to prove the theorem.

(i) $0<\gamma<\frac{a}{3}$. From case (i) of Theorem 2, we know that $\gamma(a-3\gamma)\geqslant 0$ and $G*\left(\frac{(a-3\gamma)\gamma}{2}u^2\right)\geqslant 0$. Thanks to Holger's inequality and Yong's inequality, we have

$$\left|\int_S u_x^2 u^2 \mathrm{d}x\right| \leqslant \| u \|_{L^\infty}^2 \int_S u_x^2 \mathrm{d}x \leqslant \frac{\mathrm{e}+1}{2(\mathrm{e}-1)}\| u \|_{H^1}^4 \leqslant \frac{\mathrm{e}+1}{2(\mathrm{e}-1)}\| u_0 \|_{H^1}^4 \mathrm{e}^{4\beta T} \tag{14}$$

$$\left|\int_S u_x^2 u^2 \mathrm{d}x\right| \leqslant \| u \|_{L^\infty} \int_S u_x^2 \mathrm{d}x \leqslant \sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\| u \|_{H^1}^3 \leqslant \sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\| u_0 \|_{H^1}^3 \mathrm{e}^{3\beta T} \tag{15}$$

$$\left|\int_S u_x^2 G * u\mathrm{d}x\right| \leqslant \| G * u \|_{L^\infty} \int_S u_x^2 \mathrm{d}x \leqslant \sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\| u \|_{H^1}^3 \leqslant \sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\| u_0 \|_{H^1}^3 \mathrm{e}^{3\beta T} \tag{16}$$

and

$$\left|\int_S u_x^2 G * u_x \mathrm{d}x\right| \leqslant \| G * u_x \|_{L^\infty} \int_S u_x^2 \mathrm{d}x \leqslant \frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\| u \|_{H^1}^3 \leqslant$$

$$\frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\| u_0 \|_{H^1}^3 \mathrm{e}^{3\beta T} \tag{17}$$

Using equations (14)~(17) and equation (13), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \gamma u_x^3\,\mathrm{d}x \leqslant -\frac{3}{2\|u_0\|_{H^1}^2 \mathrm{e}^{2\beta T}}\left(\int_S \gamma u_x^3\,\mathrm{d}x+\beta\|u_0|\,\mathrm{e}^{2\beta T}\right)^2+\frac{3\beta^2}{2}\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}+$$

$$\frac{3\gamma(a-2\gamma)(\mathrm{e}+1)}{4(\mathrm{e}-1)}\|u_0\|_{H^1}^4\mathrm{e}^{4\beta T}+\left(12\tilde{\omega}\gamma+3\beta\gamma\frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\|u_0\|_{H^1}^3\mathrm{e}^{3\beta T} \tag{18}$$

Setting
$$m(t)=\int_S \gamma u_x^3\,\mathrm{d}x-\beta\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}$$

and
$$K=\frac{3\beta^2}{2}\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}+\frac{3\gamma(a-2\gamma)(\mathrm{e}+1)}{4(\mathrm{e}-1)}\|u_0\|_{H^1}^4\mathrm{e}^{4\beta T}+\left(12\tilde{\omega}\gamma+3\beta\gamma\frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\|u_0\|_{H^1}^3\mathrm{e}^{3\beta T}$$

We have
$$\frac{\mathrm{d}}{\mathrm{d}t}m(t)\leqslant-\frac{3}{2\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}}m^2(t)+K \tag{19}$$

From Lemma 4, if $m(0)<-\sqrt{\frac{2K}{3}}\|u_0\|_{H^1}\mathrm{e}^{\beta T}$, then there exists T such that $\lim_{t\uparrow T}m(t)=-\infty$. Applying Theorem 2, the solution u blows up in finite time.

(ii) $\frac{a}{3}<\gamma\leqslant a$. From case (ii) of Theorem 2, we have $\gamma(3\gamma-a)\geqslant 0$ and $G*\frac{\gamma(3\gamma-a)}{4}u_x^2\geqslant 0$. Using equations (14)～(17) and equation (13), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \gamma u_x^3\,\mathrm{d}x \leqslant -\frac{3}{2\|u_0\|_{H^1}^2 \mathrm{e}^{2\beta T}}\left(\int_S \gamma u_x^3\,\mathrm{d}x+\beta\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}\right)^2+\frac{3\beta^2}{2}\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}+\frac{3\gamma(a-\gamma)(\mathrm{e}+1)}{8(\mathrm{e}-1)}\|u_0\|_{H^1}^4\mathrm{e}^{4\beta T}+$$

$$\left(12\tilde{\omega}\gamma+3\beta\gamma\frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\|u_0\|_{H^1}^3\mathrm{e}^{3\beta T} \tag{20}$$

Setting
$$m(t)=\int_S \gamma u_x^3\,\mathrm{d}x+\beta\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}$$

and
$$K=\frac{3\beta^2}{2}\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}+\frac{3\gamma(a-\gamma)(\mathrm{e}+1)}{8(\mathrm{e}-1)}\|u_0\|_{H^1}^4\mathrm{e}^{4\beta T}+\left(12\tilde{\omega}\gamma+3\beta\gamma\frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\|u_0\|_{H^1}^3\mathrm{e}^{3\beta T}$$

We have
$$\frac{\mathrm{d}}{\mathrm{d}t}m(t)\leqslant-\frac{3}{2\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}}m^2(t)+K \tag{21}$$

Similar to the proof in case (i), we derive that the corresponding solution will blow up in finite time.

(iii) $\gamma>a$ or $\gamma<0$, note that $\frac{(a-\gamma)\gamma}{2}<0$.

Using equations (14)～(17) and equation (13), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \gamma u_x^3\,\mathrm{d}x \leqslant -\frac{3}{2\|u_0\|_{H^1}^2 \mathrm{e}^{2\beta T}}\left(\int_S \gamma u_x^3\,\mathrm{d}x+\beta\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}\right)^2+\frac{3\beta^2}{2}\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}+\frac{3\gamma(a-\gamma)(\mathrm{e}+1)}{8(\mathrm{e}-1)}\|u_0\|_{H^1}^4\mathrm{e}^{4\beta T}+$$

$$\left(12\tilde{\omega}|\gamma|+3\beta|\gamma|\frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\|u_0\|_{H^1}^3\mathrm{e}^{3\beta T} \tag{22}$$

Setting
$$m(t)=\int_S \gamma u_x^3\,\mathrm{d}x+\beta\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}$$

and
$$K=\frac{3\beta^2}{2}\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}+\frac{3\gamma(a-\gamma)(\mathrm{e}+1)}{8(\mathrm{e}-1)}\|u_0\|_{H^1}^4\mathrm{e}^{4\beta T}+\left(12\tilde{\omega}|\gamma|+3\beta|\gamma|\frac{\cosh\frac{1}{2}-1}{\sinh\frac{1}{2}}\right)\sqrt{\frac{\mathrm{e}+1}{2(\mathrm{e}-1)}}\|u_0\|_{H^1}^3\mathrm{e}^{3\beta T}$$

We have
$$\frac{\mathrm{d}}{\mathrm{d}t}m(t)\leqslant-\frac{3}{2\|u_0\|_{H^1}^2\mathrm{e}^{2\beta T}}m^2(t)+K \tag{23}$$

Similar to the proof in case (i), we conclude that the corresponding solution of equation (3) will blow up in finite time. It completes the proof of Theorem 2.

Remark 2 Theorems 2 improve considerably recent results in [1, 17].

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一个推广的周期非线性色散波方程的爆破解

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摘要：使用经典的数学技巧研究了一个推广的周期非线性色散波方程的柯西问题。通过使用 Kato 半群理论，获得了这个方程局部解的存在唯一性。在关于初值的适合条件下，得到了这个方程的一个精确爆破图景（即解在有限时间爆破当且仅当$\lim\limits_{t\uparrow T}\sup\{\sup\limits_{x\in S}|\gamma u_x(t,x)|\}=+\infty$）和一个爆破结果（即解爆破的一个充分条件）。

关键词：非线性色散波方程；局部适定性；爆炸图景；爆破解；Camassa-Holm 方程；ROD 方程

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