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Uniqueness of Entire Functions Sharing a Polynomial*

WU Chun

(College of Mathematics Science, Chongqing Normal University, Chongqing 401331, China)

Abstract: In this paper, we study the uniqueness of entire functions sharing a nonzero polynomials, and prove that if $f(z)$ and $g(z)$ be two transcendental entire functions, and let n,k and l be three positive integers satisfying $5l>4n+5k+7$. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share P IM, where P is a nonzero polynomial with deg $P\leqslant 5$, then $f=\lambda_1 e^{\lambda Q(z)}+c$, $g=\lambda_2 e^{-\lambda Q(z)}+c$, or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $Q(z)=\int_0^z p(z)\mathrm{d}z$, $\lambda_1,\lambda_2,\lambda$ and c are constants such that $(\lambda_1\lambda_2)^n(n\lambda)^2=-1$, and $R(\omega_1,\omega_2)=L(\omega_1)-L(\omega_2)$. Moreover, we also obtain the results that $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share the fixed point IM or CM.

Key words: uniqueness; entire functions; sharing a polynomial; dierential polynomial

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1 Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We will use the standard notations of Nevanlinna's value distribution theory such as $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $m(r,f)$ and so on, as explained in Hayman[1], Yang[2] and Yi and Yang[3].

Let a be a finite complex number, and k be a positive integer. We denote by $N_{k)}\left(r,\frac{1}{f-a}\right)$ the counting function for zeros of $f-a$ with multiplicity $\leqslant k$, and by $\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r,\frac{1}{f-a}\right)$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}\left(r,\frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Set $N_k\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\overline{N}_{(k}\left(r,\frac{1}{f-a}\right)$.

We define $\Theta(a,f)=1-\varlimsup\limits_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f-a}\right)}{T(r,f)}$, and $\delta_k(a,f)=1-\varlimsup\limits_{r\to\infty}\frac{N_k\left(r,\frac{1}{f-a}\right)}{T(r,f)}$.

Let f and g be two non-constant meromorphic functions, a be a finite complex number. The definitions of f,g sharing the value a CM (or IM), $\overline{N}_L\left(r,\frac{1}{f-a}\right)$, $N_0\left(r,\frac{1}{F'}\right)$ and $\overline{N}_{f>k}\left(r,\frac{1}{g-1}\right)$. We refer the reader to [3-5].

In [6], Fang got the following results.

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First author biography: WU Chun, male, lechure, master, engaged in the study of value distribution theory, E-mail: wuchun@cqnu.edu.cn

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作者简介:吴春,男,讲师,硕士,研究方向为值分布理论,E-mail:wuchun@cqnu.edu.cn

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Theorem A Let f and g be two non-constant entire functions, and let n,k be two positive integers with $n>2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z)=c_1e^{cz}, g(z)=c_2e^{-cz}$, where c_1,c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$ or $f=tg$ for a constant t such that $t^n=1$.

Theorem B Let f and g be two non-constant entire functions, and let n,k be two positive integers with $n>2k+8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f=g$.

In 2008, J. F. Chen, X. Y. Zhang[7] improved the above result and obtained the following result.

Theorem C Let f and g be two non-constant entire functions, and let n,k be two positive integers with $n>5k+7$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 IM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1,c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$ or $f=tg$ for a constant t such that $t^n=1$.

Theorem D Let f and g be two non-constant entire functions, and let n,k be two positive integers with $n>5k+13$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 IM, then $f=g$.

In 2008, X. Y. Zhang, J. F. Chen and W. C. Lin[8] extended the above result by proving the following result.

Theorem E Let f and g be two entire functions; let n,m and k be three positive integers with $n>3m+2k+5$, and let $P(z)=a_mz^m+a_{m-1}z^{m-1}+\cdots+a_1z+a_0$ or $P(z)=C$, where $a_0\neq 0,a_1,\cdots,a_{m-1},a_m\neq 0$, $C\neq 0$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 CM, then either $f(z)=\lambda_1e^{\lambda z}$, $g(z)=\lambda_2e^{-\lambda z}$, where $\lambda_1,\lambda_2,\lambda$ are three constants satisfying $(-1)^k(\lambda_1\lambda_2)^n(n\lambda)^{2k}C^2=1$, or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)=0$, where $R(\omega_1,\omega_2)=\omega_1^np(\omega_1)-\omega_2^np(\omega_2)$.

In this paper we always use $L(z)$ denoting a arbitrary polynomial of degree n, i. e.

$$L(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0=a_n(z-c)^{l_1}(z-c_2)^{l_2}\cdots(z-c_s)^{l_s} \tag{1}$$

where $a_i(i=0,1,\cdots,n)$, $a_n\neq 0$ and $c_j(j=1,2,\cdots,s)$ are finite complex number constants, and $c,c_2,\cdots,c_s$ are all the distinct zeros of $L(z)$, $l_1,l_2,\cdots,l_s,s,n$ are all positive integers satisfying

$$l_1+l_2+\cdots+l_s=n,\text{and let } l=\max\{l_1,l_2,\cdots,l_s\} \tag{2}$$

Corresponding to the above results, some authors considered the uniqueness problems of entire functions that have fixed points, see M. L. Fang and H. Qiu[9], W. C. Lin and H. X. Yi[10], J. Dou, X. G. Qi and L. Z. Yang[11].

In this paper, we consider the existence of solutions of $[L(f)]^{(k)}-P$ and the corresponding uniqueness theorems, and we obtain the following results which generalize the above theorems.

Theorem 1 Let f be a transcendental entire function. When $n>k+s$, then $[L(f)]^{(k)}=P$ has infinitely many solutions, where $P\not\equiv 0$ is a polynomial.

Remark 1 It is easy to see that a polynomial $Q(z)-P(z)$ has exactly $\max\{m,n\}$ solutions (counting multiplicities), where $\deg Q=m$, $\deg P=n$, but a transcendental entire function may have no solution. For example, let $f(z)=e^{\alpha(z)}+P(z)$, then function $f(z)-P(z)$ has no any solution, where $\alpha(z)$ is an entire function.

Theorem 2 $f(z)$ and $g(z)$ be two transcendental entire functions, and let n,k and l be three positive integers satisfying $5l>4n+5k+7$. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share P IM, where P is a nonzero polynomial with $\deg P\leqslant 5$, then $f=\lambda_1e^{\lambda Q(z)}+c, g=\lambda_2e^{-\lambda Q(z)}+c$, or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $Q(z)=\int_0^z p(z)\mathrm{d}z$, $\lambda_1,\lambda_2,\lambda$ and c are constants such that $(\lambda_1\lambda_2)^n(n\lambda)^2=-1$, and $R(\omega_1,\omega_2)=L(\omega_1)-L(\omega_2)$.

Remark 2 When $l=n, l=n-1$, respectively, and $c=0, P(z)=1$, from Theorem 2 we can easily get Theorem C,D.

Corollary 1 Let $f(z)$ and $g(z)$ be two transcendental entire functions, and let n,k and l be three positive integers satisfying $5l>4n+5k+7$. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ have the same fixed points ignoring multiplicities, then $f=\lambda_1e^{\lambda z^2}+c, g=\lambda_2e^{-\lambda z^2}+c$, or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $\lambda_1,\lambda_2,\lambda$ and c are constants, satisfying $4(\lambda_1\lambda_2)^n(\lambda)^2=-1$, and $R(\omega_1,\omega_2)=L(\omega_1)-L(\omega_2)$.

Theorem 3 Let $f(z)$ and $g(z)$ be two transcendental entire functions, and let n,k and l be three positive integers satisfying $l>\frac{n}{2}+k+2$. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share P CM, where P is a nonzero polynomial with deg $P\leqslant 5$, then $f=\lambda_1 e^{\lambda Q(z)}+c$, $g=\lambda_2 e^{-\lambda Q(z)}+c$, or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $Q(z)=\int_0^z p(z)\mathrm{d}z$, $\lambda_1,\lambda_2,\lambda$ and c are constants such that $(\lambda_1\lambda_2)^n(n\lambda)^2=-1$, and $R(\omega_1,\omega_2)=L(\omega_1)-L(\omega_2)$.

Remark 3 When $l=n$, $c=0$ and $P(z)=1$, from Theorem 3 we can easily get Theorem A; When $l=n-1$, $l=n-m$, respectively, and $c=0$, $P(z)=1$, Theorem 3 improves Theorem B, E.

Corollary 2 Let $f(z)$ and $g(z)$ be two transcendental entire functions, and let n, k and l be three positive integers satisfying $l>\frac{n}{2}+k+2$. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ have the same fixed points counting multiplicities, then $f=\lambda_1 e^{\lambda z^2}+c$, $g=\lambda_2 e^{-\lambda z^2}+c$, or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where λ_1, λ_2, λ and c are constants, satisfying $4(\lambda_1\lambda_2)^n(\lambda)^2=-1$, and $R(\omega_1,\omega_2)=L(\omega_1)-L(\omega_2)$.

Remark 4 If $L(f)\equiv L(g)$, we obtain $a_n f^n+a_{n-1}f^{n-1}+\cdots+a_1 f\equiv a_n g^n+a_{n-1}g^{n-1}+\cdots+a_1 g$.

Let $h=\frac{f}{g}$. If h is a constant, then substituting $f=gh$ into above equation we deduce $a_n g^n(h^n-1)+a_{n-1}g^{n-1}(h^{n-1}-1)+\cdots+a_1 g(h-1)\equiv 0$, which implies $h^d=1$, $d=(n,\cdots,n-i,\cdots,1)$, $a_{n-i}\neq 0$ for some $i=0,1,\cdots,n-1$. Thus $f\equiv tg$ for a constant t such that $t^d=1$. If h is not a constant, then we know by above equation that f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\omega_1,\omega_2)=L(\omega_1)-L(\omega_2)$.

Remark 5 Moreover, let $L(z)$ is a generic polynomial of degree at least 5. Then from the equation $L(f)\equiv L(g)$, one can conclude that $f\equiv g$ and no other nonconstant meromorphic solutions f and g. In [12], Yang-Hua exhibits some classes of such polynomials. And some related definitions and results, we refer the reader to [13-14].

2 Some lemmas

Lemma 1[1] Let $f(z)$ be a nonconstant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r,a_i)=S(r,f)(i=1,2)$. Then $T(r,f)=\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)+S(r,f)$.

Lemma 2[15] Let $f(z)$ be a non-constant meromorphic function, let $k(\geqslant 1)$ be a positive integer and let $\varphi(\neq 0,\infty)$ be a small function of f. Then

$$T(r,f)\leqslant\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-\varphi}\right)-N\left(r,\frac{1}{(f^{(k)}/\varphi)'}\right)+S(r,f) \quad (3)$$

Lemma 3[16] Let $a_n(\neq 0)$, a_{n-1}, $\cdots$, a_0 be constants, and let f be a nonconstant meromorphic function, then $T(r,a_n f^n+a_{n-1}f^{n-1}+\cdots+a_1 f+a_0)=nT(r,f)$.

Lemma 4[17] Let $f(z)$ be a non-constant meromorphic function, s, k be two positive integers. Then $N_s\left(r,\frac{1}{f^{(k)}}\right)\leqslant k\overline{N}(r,f)+N_{s+k}\left(r,\frac{1}{f}\right)+S(r,f)$.

Clearly, $\overline{N}\left(r,\frac{1}{f^{(k)}}\right)=N_1\left(r,\frac{1}{f^{(k)}}\right)$.

Lemma 5[4] Let $f(z)$ be a nonconstant entire function, n, k a positive integer, and let c be a nonzero finite complex number. Then $T(r,f)\leqslant N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-c}\right)-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)$, where $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\neq 0$.

Lemma 6[4] Let $f(z)$ be a nonconstant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)}\not\equiv 0$, then $N\left(r,\frac{1}{f^{(k)}}\right)\leqslant k\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+S(r,f)$.

Lemma 7[5]　Let f,g share $(1,0)$. Then i) $\overline{N}_{f>1}\left(r,\frac{1}{g-1}\right)\leqslant\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)-N_0\left(r,\frac{1}{f'}\right)+S(r,f)$; ii) $\overline{N}_{g>1}\left(r,\frac{1}{f-1}\right)\leqslant\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r,g)$.

Lemma 8　Let f and g be two non-constant entire functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share a nonzero polynomial $p(z)$ IM, and if $\Delta=\delta_{k+2}(0,g)+\delta_{k+2}(0,f)+\delta_{k+1}(0,f)+2\delta_{k+1}(0,g)>4$, then $\frac{p}{f^{(k)}-p}=\frac{bg^{(k)}+(a-b)p}{g^{(k)}-p}$, where a,b are two constants.

Proof　Let

$$F=\frac{f^{(k)}}{p},G=\frac{g^{(k)}}{p}\tag{4}$$

and let

$$h=\left[\frac{F''}{F}-\frac{2F'}{F-1}\right]-\left[\frac{G''}{G}-\frac{2G'}{G-1}\right]\tag{5}$$

Clearly $m(r,h)=S(r,f)+S(r,g)$. Since $f^{(k)}-p$ and $g^{(k)}-p$ share 0 IM, a simple computation on local expansions shows that $h(z_0)=0$, if z_0 is a common simple zero of $f^{(k)}-p$ and $g^{(k)}-p$. Next we consider two cases: $h\not\equiv0$ and $h\equiv0$.

If $h\not\equiv0$, then

$$N_{11}\left(r,\frac{1}{F-1}\right)=N_{11}\left(r,\frac{1}{G-1}\right)\leqslant\overline{N}\left(r,\frac{1}{h}\right)+O(1)\leqslant N(r,h)+S(r,f)+S(r,g)\tag{6}$$

Let $z_0\notin\{z:p(z)=0\}$ be a common simple zero of $f^{(k)}-p$ and $g^{(k)}-p$. Then it follows from (4) that z_0 is a common simple zero of $F-1$ and $G-1$, by calculating we get $h(z_0)=0$.

Let $z_1\notin\{z:p(z)=0\}$ be a simple pole of F, then by calculating we see that $\frac{F''}{F'}-\frac{2F'}{F-1}$ is analytic at z_1. Similarly, if $z_2\notin\{z:p(z)=0\}$ be a simple pole of G, we see that $\frac{G''}{G'}-\frac{2G'}{G-1}$ is analytic at z_2. Thus, we see that the poles of h come from those common zeros of $F-1$ and $G-1$ such that the multiplicity of each such zero of $F-1$ is different from that of the same zero of $G-1$, those zeros of F' that are not the zeros of $F(F-1)$, those zeros of G' that are not the zeros of $G(G-1)$, those zeros of F with their multiplicities$\geqslant2$, those zeros of G with their multiplicities$\geqslant2$, and each point is counted only once. Therefore,

$$N(r,h)\leqslant\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+O(\log r)\tag{7}$$

Suppose that $z_0\notin\{z:p(z)=0\}$ is a zero of F with its multiplicity $\tau\geqslant k+2$, then it follows from (4) that z_0 is a zero of F' with its multiplicity $\tau-k-1\geqslant1$. Thus from (4) and Lemma 2 we get

$$\begin{aligned}T(r,g)&\leqslant N\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{G-1}\right)-N\left(r,\frac{1}{G'}\right)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-\\&N_0\left(r,\frac{1}{G'}\right)+O(\log r)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f)\end{aligned}\tag{8}$$

Since

$$\overline{N}\left(r,\frac{1}{G-1}\right)=N_{11}\left(r,\frac{1}{G-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_{G>1}\left(r,\frac{1}{F-1}\right)\tag{9}$$

Thus we deduce from (4)～(9) that

$$\begin{aligned}T(r,g)&\leqslant N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+N_0\left(r,\frac{1}{F'}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\\&\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_{G>1}\left(r,\frac{1}{F-1}\right)+S(r,f)+S(r,g)\end{aligned}\tag{10}$$

From the definition of $N_0\left(r,\frac{1}{F'}\right)$, we see that $N_0\left(r,\frac{1}{F'}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+N_{(2}\left(r,\frac{1}{F}\right)-\overline{N}_{(2}\left(r,\frac{1}{F}\right)\leqslant N\left(r,\frac{1}{F'}\right)$.

The above inequality and Lemma 6 give

$$N_0\left(r,\frac{1}{F'}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)\leqslant N\left(r,\frac{1}{F'}\right)-N_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)\leqslant$$

$$N\left(r,\frac{1}{F}\right)-N_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+S(r,f)\leqslant\overline{N}\left(r,\frac{1}{F}\right)+S(r,f) \tag{11}$$

From (4) and Lemma 4 we obtain

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leqslant N\left(r,\frac{1}{F-1}\right)-\overline{N}\left(r,\frac{1}{F-1}\right)\leqslant N\left(r,\frac{F'}{F}\right)\leqslant\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)\leqslant$$

$$\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+O(\log r)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{f}\right)+S(r,f) \tag{12}$$

Similarly

$$\overline{N}_L\left(r,\frac{1}{G-1}\right)\leqslant N_{k+1}\left(r,\frac{1}{g}\right)+S(r,g) \tag{13}$$

From (10)～(13), Lemma 4 and Lemma 7 we get

$$T(r,g)\leqslant N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)+N_{k+1}\left(r,\frac{1}{f}\right)+$$

$$N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{G}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f)+S(r,g)\leqslant$$

$$N_{k+2}\left(r,\frac{1}{g}\right)+N_{k+2}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{f}\right)+2N_{k+1}\left(r,\frac{1}{g}\right)-O(\log r)+S(r,f)+S(r,g)\leqslant$$

$$N_{k+2}\left(r,\frac{1}{g}\right)+N_{k+2}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{f}\right)+2N_{k+1}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g) \tag{14}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f)\leqslant T(r,g)$ for $r\in I$. Hence

$$T(r,g)\leqslant\{[1-\delta_{k+2}(0,g)]+[1-\delta_{k+2}(0,f)]+[1-\delta_{k+1}(0,f)]+$$

$$2[1-\delta_{k+1}(0,g)]+\varepsilon\}T(r,g)+S(r,g) \tag{15}$$

for $r\in I$ and $0<\varepsilon<\Delta-4$, that is $\{\Delta-4-\varepsilon\}T(r,g)\leqslant S(r,g)$. i.e., $\Delta-4\leqslant 0$, which is a contradiction to our hypotheses $\Delta>4$.

Hence, we get $h\equiv 0$. Therefore by (5), we have $\frac{F''}{F}-\frac{2F'}{F-1}=\frac{G''}{G}-\frac{2G'}{G-1}$.

By integrating two sides of the above equality, we obtain $\frac{p}{f^{(k)}-p}=\frac{bg^{(k)}+(a-b)p}{g^{(k)}-p}$, where $a(\neq 0)$ and b are constants. The proof of the Lemma is completed.

Lemma 9 Let f and g be two non-constant entire functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share a nonzero polynomial $p(z)$ CM, and if $\Delta=\delta_{k+2}(0,g)+\delta_{k+2}(0,f)>1$, then $\frac{p}{f^{(k)}-p}=\frac{bg^{(k)}+(a-b)p}{g^{(k)}-p}$, where a,b are two constants.

Proof Since $f^{(k)}$ and $g^{(k)}$ share the value $p(z)$ CM, we have $\overline{N}_L\left(r,\frac{1}{F-1}\right)=\overline{N}_L\left(r,\frac{1}{G-1}\right)=0$. Proceeding as in the proof of Lemma 8, we obtain conclusion of Lemma 9.

By using the main idea is from [11,18], we easily obtain the following lemma.

Lemma 10 Let f and g be two non-constant entire functions, and let $P(z)\not\equiv 0$ be a polynomial. If $[f^n]^{(k)}[g^n]^{(k)}\equiv p^2$ and $\deg p\leqslant 5$, then $f=b_1e^{bQ}$, $g=b_2e^{-bQ}$, where b_1,b_2,b are three constants satisfying $(b_1b_2)^n(nb)^2=-1$, Q is a polynomial satisfying $Q=\int_0^z p(\eta)\mathrm{d}\eta$.

3 Proof of theorems

3.1 Proof of theorem 1

Because f is a transcendental entire, we get $T(r,p)=o(T(r,f))$. Suppose that $z_0\notin\{z:p(z)=0\}$ is a zero of $L(f)$ with its multiplicity $l\geqslant k+2$, then z_0 is a zero of $\frac{[L(f)]^{(k)}}{P'}$ with its multiplicity $l-k-1\geqslant 1$. From Lemma 2 and Lemma 3, we have

$$nT(r,f)=T(r,L(f))+S(r,f)\leqslant N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(f)^{(k)}-p}\right)-N\left(r,\frac{1}{(L(f)(k)/p)'}\right)+S(r,f)\leqslant$$
$$N_{k+1}\left(r,\frac{1}{L(f)}\right)+\overline{N}\left(r,\frac{1}{L(f)^{(k)}-p}\right)-N_0\left(r,\frac{1}{(L(f)^{(k)}/p)'}\right)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{(f-c_1)^{l_1}}\right)+\cdots+$$
$$N_{k+1}\left(r,\frac{1}{(f-c_s)^{l_s}}\right)+\overline{N}\left(r,\frac{1}{L(f)^{(k)}-p}\right)+S(r,f)\leqslant(k+s)T(r,f)+\overline{N}\left(r,\frac{1}{L(f)^{(k)}-p}\right)+S(r,f)$$

Thus, we get $(n-k-s)T(r,f)\leqslant\overline{N}\left(r,\frac{1}{L(f)^{(k)}-p}\right)+S(r,f)$. Noting that $n>k+s$, we get $L(f)^{(k)}=p$ has infinitely many solutions.

3.2 Proof of theorem 2

Let $L(z)$ and l be given by (1) and (2), respectively. Without loss of generality, we suppose that $a_n=1$, $l=l_1$, and $c=c_1$. We get

$$\delta_{k+1}(0,L(f))=1-\overline{\lim_{r\to\infty}}\frac{N_{k+1}(r,1/L(f))}{T(r,L(f))}\geqslant 1-\overline{\lim_{r\to\infty}}\frac{\sum_{j=2}^{s}N_{k+1}(r,1/(f-c_j)^{l_j})+N_{k+1}(r,1/(f-c)^{l})}{nT(r,f)}\geqslant$$
$$1-\overline{\lim_{r\to\infty}}\frac{\sum_{j=2}^{s}(s-1)T(r,f)+(k+1)T(r,f)+S(r,f)}{nT(r,f)}\geqslant 1-\frac{s+k}{n}\geqslant\frac{l-k-1}{n}$$

Similarly, we have $\delta_{k+1}(0,L(g))\geqslant 1-\frac{s+k}{n}\geqslant\frac{l-k-1}{n}$. Because $5l\geqslant 4n+5k+7$, we obtain $\delta_{k+2}(0,L(g))+\delta_{k+2}(0,L(f))+\delta_{k+1}(0,L(f))+2\delta_{k+1}(0,L(g))\geqslant 4$.

By Lemma 8, we can obtain

$$\frac{p}{L(f)^{(k)}-p}=\frac{bL(g)^{(k)}+(a-b)p}{L(g)^{(k)}-p}\tag{16}$$

Next, we consider the following three cases.

Case 1 $b\neq 0, a=b$. Then from (16) we have

$$\frac{p}{L(f)^{(k)}-p}=\frac{bL(g)^{(k)}}{L(g)^{(k)}-p}\tag{17}$$

Case 1.1 If $b=-1$, then it follows from (17) that $[L(f)]^{(k)}[L(g)]^{(k)}\equiv p^2$.
That is

$$[(f-c)^{l}(f-c_2)^{l_2}\cdots(f-c_s)^{l_s}]^{(k)}[(g-c)^{l}(g-c_2)^{l_2}\cdots(g-c_s)^{l_s}]^{(k)}\equiv p^2\tag{18}$$

Case 1.1.1 When $s=1$, we can rewrite (18) $[(f-c)^n]^{(k)}\cdot[(g-c)^n]^{(k)}\equiv p^2$. By using Lemma 10, we can easily obtain $f=b_1e^{bQ}+c, g=b_2e^{-bQ}+c$, where b_1, b_2, b are three constants satisfying $(b_1b_2)^n(nb)^2=-1, Q$ is a polynomial satisfying $Q=\int_0^z p(\eta)\mathrm{d}\eta$.

Case 1.1.2 When $s\geqslant 2$, we notice that $5l>4n+5k+7$, hence $l>5k+7$. Suppose that $z_0\notin\{z:p(z)=0\}$ is a l-fold zero of $f-c$, we know that z_0 must be a $l-k$-fold zero of $[(f-c)^{l}(f-c_2)^{l_2}\cdots(f-c_s)^{l_s}]^{(k)}$. Noting that g is an entire function, it follows from (18), which is a contradiction. Hence $f-c\neq 0, g-c\neq 0$. So we get $f=e^{\alpha(z)}+c$, where $\alpha(z)$ is a nonconstant entire function. Thus we have

$$[f^i]^{(k)}=[(e^{\alpha(z)}+c)^i]^{(k)}=p_i(\alpha',\alpha'',\cdots,\alpha^{(k)})e^{i\alpha}, i=1,2,\cdots,n\tag{19}$$

where $p_i(i=1,2,\cdots,n)$ is differential polynomials about $\alpha',\alpha'',\cdots,\alpha^{(k)}$. Obviously, $p_i\not\equiv 0$, $T(r,p_i)=S(r,f)$ $(i=1,2,\cdots,n)$, we get from (18) to (19) that $N\left(r,\frac{1}{p_ne^{(n-1)\alpha}+\cdots+p_1}\right)=S(r,f)$.

According to Lemma 1 and Lemma 5 and $f=e^{\alpha(z)}+c$, we get

$$(n-1)T(r,f-c)=T(r,p_n e^{(n-1)\alpha}+\cdots+p_1)+S(r,f)\leqslant\overline{N}\left(r,\frac{1}{p_n e^{(n-1)\alpha}+\cdots+p_1}\right)+$$
$$\overline{N}\left(r,\frac{1}{p_n e^{(n-1)\alpha}+\cdots+p_2 e^{\alpha}}\right)+S(r,f)\leqslant\overline{N}\left(r,\frac{1}{p_n e^{(n-2)\alpha}+\cdots+p_2}\right)+S(r,f)\leqslant(n-2)T(r,f-c)+S(r,f)$$

which is a contradiction.

Case 1.2 If $a=b\neq-1$, then (17) that can be written as

$$\frac{L(g)^{(k)}}{p}=\frac{-1}{b}\cdot\frac{1}{L(f)^{(k)}/p-(1+b)/b} \tag{20}$$

From (20), we get

$$\overline{N}\left(r,\frac{1}{L(f)^{(k)}/p-(1+b)/b}\right)=\overline{N}(r,g)+O(\log r)=S(r,f) \tag{21}$$

by (21) and Lemma 2, we get

$$nT(r,f)=T(r,L(f))+O(1)\leqslant N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(f)^{(k)}/p-(1+b)/b}\right)-N\left(r,\frac{1}{(L(f)^{(k)}/p)'}\right)+S(r,f)\leqslant$$
$$N_{k+1}\left(r,\frac{1}{L(f)}\right)+\overline{N}(r,g)+O(\log r)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{L(f)}\right)+S(r,f)\leqslant N_{k+1}\left(r,\frac{1}{(f-c)^{l}}\right)+$$
$$N_{k+1}\left(r,\frac{1}{(f-c_2)^{l_2}\cdots(f-c_s)^{l_s}}\right)+S(r,f)\leqslant(k+s)T(r,f)+S(r,f)\leqslant(k+n-l+1)T(r,f)+S(r,f)$$

which is a contradiction, because $5l>4n+5k+7$.

Case 2 $b\neq0,a\neq b$. We discuss the following we subcases:

Case 2.1 Suppose that $b=-1$, then $a\neq0$ and (16) can be rewritten as

$$\frac{L(f)^{(k)}}{p}=\frac{a}{a+1-L(g)^{(k)}/p} \tag{22}$$

From (22) we get

$$\overline{N}\left(r,\frac{1}{a+1-L(g)^{(k)}/p}\right)=\overline{N}(r,f)+O(\log r)=S(r,g) \tag{23}$$

From (23), Lemma 2 and Lemma 4, we get $nT(r,g)=T(r,L(g))+O(1)\leqslant N_{k+1}\left(r,\frac{1}{L(g)}\right)+S(r,g)$.

Next, by using the argument as in case 1.2, we get a contradiction.

Case 2.2 Suppose that $b\neq-1$, then (16) be rewritten as

$$\frac{L(f)^{(k)}}{p}-\frac{b+1}{b}=\frac{-a}{b^2}\cdot\frac{1}{L(g)^{(k)}/p+(a-b)/b} \tag{24}$$

From(24), we get

$$\overline{N}\left(r,\frac{1}{L(f)^{(k)}/p-(b+1)/b}\right)=\overline{N}(r,g)+O(\log r) \tag{25}$$

From (25), Lemma 2, Lemma 4 and in the same manner as in case 1.2, we can get a contradiction.

Case 3 $b=0,a\neq0$. Then (16) can be rewritten as

$$L(g)=aL(f)+(1-a)p_1(z) \tag{26}$$

where $p_1(z)$ is a polynomial with its deg $p_1\geqslant k+1$. If $a\neq1$, then $(1-a)p_1(z)\not\equiv0$. This together with (26) and Lemma 1, we get

$$nT(r,g)=T(r,L(g))+O(1)\leqslant\overline{N}\left(r,\frac{1}{L(g)}\right)+\overline{N}\left(r,\frac{1}{L(f)}\right)+S(r,g)\leqslant$$
$$\sum_{i=1}^{s}\overline{N}\left(r,\frac{1}{g-c_i}\right)+\sum_{i=1}^{s}\overline{N}\left(r,\frac{1}{f-c_i}\right)+S(r,g)\leqslant s[T(r,g)+T(r,f)]+S(r,g) \tag{27}$$

because $n=l+l_2+\cdots+l_s$, we get $n-l=l_2+\cdots+l_s\geqslant s-1$, i.e., $n-l\geqslant s-1$. From $5l>4n+5k+7$, we have $l-1>4(n-l)+5k+6>4(s-1)+5k+6$, so $n-s\geqslant l-1>4(s-1)+5k+6$, i.e., $n-s>4(s-1)+5k+6$, thus, $s<\frac{n-5k-2}{5}$. Thus

$$nT(r,g)<\frac{n-5k-2}{5}[T(r,g)+T(r,f)]+S(r,g) \tag{28}$$

On the other hand, from (26) and Lemma 3, we see that $T(r,g)=T(r,f)+S(r,g)$.

Substituting this into (28), we deduce that $\frac{3n+10k+4}{5}T(r,g)<S(r,g)$, which is a contradiction. Thus $a=1$, and so it follows from (26) that $L(f)=L(g)$.

Hence, this completes the proof of Theorem 2.

3.3 Proof of theorem 3

By using lemma 9 and the condition $l>\frac{n}{2}+k+2$, proceeding as in the proof of theorem 2, we can similarly prove theorem 3. We omit the details here.

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分担一个多项式的全纯函数的唯一性

吴 春

(重庆师范大学 数学学院，重庆 401331)

摘要：本文主要研究了全纯函数分担一个非零多项式的唯一性问题，并且得到了：若 f,g 为2个非常数的超越整函数，n,k,l 为3个正整数且满足 $5l>4n+5k+7$. 如果 $[L(f)]^{(k)}$ 与 $[L(g)]^{(k)}$ IM 分担次数小于或等于5的非零多项式 $P(z)$，则或者 $f(z)=\lambda_1 e^{\lambda Q(z)}+c$, $g(z)=\lambda_2 e^{-\lambda Q(z)}+c$，或者 $f(z)$ 与 $g(z)$ 满足代数方程 $R(f,g)\equiv 0$，这里 $Q(z)=\int_0^z p(z)\mathrm{d}z$，$\lambda_1,\lambda_2,\lambda$ 及 c 为4个常数，且满足等式 $(\lambda_1\lambda_2)^n(n\lambda)^2=-1$，并且 $R(\omega_1,\omega_2)=L(\omega_1)-L(\omega_2)$. 此外，就 $[L(f)]^{(k)}$ 与 $[L(g)]^{(k)}$ IM 或 CM 分担不动点的情形也进行了详细的研究。

关键词：唯一性；整函数；分担多项式；微分多项式

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