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Normal Families of Meromorphic Function Concerning Shared Values*

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Abstract: In this paper, we study the normality criterion concerning shared value. Let $\mathcal{F}$ be a family of meromorphic func tions defined in a domain D. Let $k, n \geqslant k+2$ be positive integers, and a be a non-zero complex number. For each pair $(f,g) \in \mathcal{F}$, if $f(f^n)^{(k)}$ and $g(g^n)^{(k)}$ share a IM, and $\overline{N}(r, 1/(f^n)^{(k)}) = S(r,f)$, then $\mathcal{F}$ is normal in D. The result improves and generalizes the theorems obtained by Zeng.

Key words: meromorphic functions; normal families; sharing values

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1 Introduction and main results

In this paper, we use the standard notations and concepts of the Nevanlinna theory[1-5]. Let D be a domain in **C**, and $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞.

Let $g(z)$ be a meromorphic function, a be a finite complex number. If $f(z)$ and $g(z)$ assume the same zeros, then we say that share a IM (ignoring multiplicity) [1].

In 2004, M. Fang and L. Zalcman[6] got the following results.

Theorem A Suppose that k is a positive integer and $a \neq 0$ is a finite complex number. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. If for each pair of functions $f, g \in \mathcal{F}$, f and g share 0, $f^{(k)}$ and $g^{(k)}$ share a IM in D, and the zeros of f are of multiplicity $\geqslant k+2$, then $\mathcal{F}$ is normal in D.

In 2012, Cuiping Zeng[7] proved the following result.

Theorem B Let k be a positive integer, $a(\neq 0)$ and b be two finite values. Let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least k and $f^{(k)}(z) = b$ when $f(z) = 0$. If for each pair of functions f and g in $\mathcal{F}$, $ff^{(k)}$ and $gg^{(k)}$ share a, then $\mathcal{F}$ is normal in D.

It is natural to ask whether Theorems B can be improved by the idea of weakened condition. In this paper, we study the problem and obtain the following theorem.

Theorem 1 Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let $k, n \geqslant k+2$ be positive integers, and a be a non-zero complex number. For each pair $(f,g) \in \mathcal{F}$, if $f(f^n)^{(k)}$ and $g(g^n)^{(k)}$ share a IM, and $\overline{N}(r, 1/(f^n)^{(k)}) = S(r,f)$, then $\mathcal{F}$ is normal in D.

Example 1 Let $D = \{z: |z| < 1\}$ and $\mathcal{F} = \{\mathrm{f_m}(z) = \mathrm{e}^{\mathrm{m}z} \mid \mathrm{m} = 1, 2, \cdots\}$ *or* $\mathcal{F} = \{f_m(z) = mz \mid m = 1, 2, \cdots\}$. Obviously, for distinct positive integers m, l, we have $f_m(f_m^n)^{(k)}$ and $g_l(g_l^n)^{(k)}$ share 0 IM. However, the families $\mathcal{F}$ are not normal at $z = 0$.

Example 1 shows that the condition $a \neq 0$ in Theorem 1 is necessary.

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2 Some Lemmas

Lemma 1 (Zalcman's Lemma)[8-9] Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ and α be a real number satisfying $-1<\alpha<1$. Then if $\mathcal{F}$ is not normal at a point $z_0\in\Delta$, there exist, for each $-1<\alpha<1$: 1) a real number r, $r<1$; 2) points z_n, $|z_n|<r$; 3) positive numbers ρ_n, $\rho_n\to 0^+$; 4) functions f_n, $f_n\in\mathcal{F}$, such that $g_n(\xi)=\dfrac{f_n(z_n+\rho_n\xi)}{\rho_n^{\alpha}}$, spherically uniformly on compact subsets of **C**, where $g(\xi)$ is a non-constant meromorphic function and $g^{\#}(\xi)\leqslant g^{\#}(0)=1$. Moreover, the order of g is not greater than 2.

Lemma 2 Let $k, n\geqslant k+2$ be positive integers and $a\neq 0$ be a finite complex number, and f be a non-constant rational meromorphic function, then $f(f^n)^{(k)}-a$ has at least two distinct zeros.

Proof Case 1 Suppose that $f(f^n)^{(k)}-a$ has exactly one zero z_0.

Case 1.1 If f^n is a non-constant polynomial. Set $f(f^n)^{(k)}-a=A(z-z_0)^l$, where A is non-zero constant, l is a positive integer and $l\geqslant n-k\geqslant 2$. Then $[f(f^n)^{(k)}]'=Al(z-z_0)^{l-1}$. Hence $f(f^n)^{(k)}(z_0)=0$, which contradicts with $(f^n)^{(k)}(z_0)=a\neq 0$. Therefore f is rational but not a polynomial.

Case 1.2 If $f(f^n)^{(k)}$ is rational but not a polynomial and has exactly one zero. We set

$$f=A\frac{(z-\alpha_1)^{m_1}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}\cdots(z-\beta_t)^{n_t}} \tag{1}$$

where A is a non-zero constant and $m_i\geqslant 1(i=1,2,\cdots,s)$, $n_j\geqslant 1(j=1,2,\cdots,t)$.

Moreover, we denote

$$m_1+m_2+\cdots+m_s=M\geqslant s, n_1+n_2+\cdots+n_t=N\geqslant t \tag{2}$$

From (1), we have

$$f^n=A^n\frac{(z-\alpha_1)^{m_1 n}\cdots(z-\alpha_s)^{m_s n}}{(z-\beta_1)^{n_1 n}\cdots(z-\beta_t)^{n_t n}} \tag{3}$$

and

$$(f^n)^{(k)}=\frac{(z-\alpha_1)^{m_1 n-k}\cdots(z-\alpha_s)^{m_s n-k}g(z)}{(z-\beta_1)^{n_1 n+k}\cdots(z-\beta_t)^{n_t n+k}} \tag{4}$$

where $g(z)$ is a polynomial, and $\deg g\leqslant k(s+t-1)$. Then

$$f(f^n)^{(k)}=\frac{(z-\alpha_1)^{m_1(n+1)-k}\cdots(z-\alpha_s)^{m_s(n+1)-k}g(z)}{(z-\beta_1)^{n_1(n+1)+k}\cdots(z-\beta_t)^{n_t(n+1)+k}}=\frac{P}{Q} \tag{5}$$

$$[f(f^n)^{(k)}]'=\frac{(z-\alpha_1)^{m_1(n+1)-k-1}\cdots(z-\alpha_s)^{m_s(n+1)-k-1}g_1(z)}{(z-\beta_1)^{n_1(n+1)+k+1}\cdots(z-\beta_t)^{n_t(n+1)+k+1}} \tag{6}$$

where $g_1(z)$ is a polynomial, and $\deg g_1\leqslant(k+1)(s+t-1)$.

Since $f(f^n)^{(k)}-a$ has exactly one zero z_0, from (5), we have

$$f(f^n)^{(k)}=a+\frac{B(z-z_0)^l}{(z-\beta_1)^{n_1(n+1)+k}\cdots(z-\beta_t)^{n_t(n+1)+k}}=\frac{P}{Q} \tag{7}$$

and

$$[f(f^n)^{(k)}]'=\frac{(z-z_0)^{l-1}g_2(z)}{(z-\beta_1)^{n_1(n+1)+k+1}\cdots(z-\beta_t)^{n_t(n+1)+k+1}} \tag{8}$$

where B is a non-constant and

$$g_2(z)=B[l-(n+1)N-kt]z^t+B_{t-1}z^{t-1}+\cdots+B_0 \tag{9}$$

in which $B_0, B_1, \cdots, B_{t-1}$ are constants.

Case 1.2.1 If $l<(n+1)N+kt$. By (7), we easily obtained that $\deg(P)=\deg(Q)$. Then, from (5), we have

$$(n+1)N+kt=\deg Q=\deg P=M(n+1)-ks+\deg g\leqslant(n+1)M-ks+k(s+t-1)\leqslant(n+1)M+kt-k$$

Hence, $(n+1)M-N(n+1)\geqslant k>0$, therefore $M>N$. On the other hand, $\alpha_i\neq z_0(i=1,2,\cdots,s)$ and $\deg g_2=t$, from (2),(6) and (8), we have $(n+1)M-(k+1)s\leqslant\deg g_2=t$. That is, $(n+1)M\leqslant(k+1)s+t\leqslant(k+1)M+N$, then $2M\leqslant M(n-k)\leqslant N$, which contradicts with $M>N$.

Case 1.2.2 If $l\geqslant(n+1)N+kt$. From (9), we have $\deg g_2\leqslant t$. It follows from the proof of above that $2M<N$. On the other hand, from (6) and (8), we have $N(n+1)+kt-1\leqslant l-1\leqslant\deg g_1\leqslant(k+1)(s+t-1)$. By (2), we have $(n+1)N\leqslant(k+1)s+t-k\leqslant(k+1)M+N-k$. This means that $(k+1)M>nN$, combining with $2M\leqslant N$, we have $2n<k+2-1\leqslant n-1$, which is impossible.

Case 2 If $f(f^n)^{(k)}-a$ has no zero.

Case 2.1 Since $n\geqslant k+2$ and f is a non-constant function. It is easily obtain that f is not a polynomial.

Case 2.2 f is rational but not a polynomial. Then we have $l=0$ for (7). Proceeding as the proof of case 1.2.1, we have a contraction.

The proof of Lemma 2 is completed.

Lemma 3 Let $k, n\geqslant k+2$ be positive integers and $a\neq 0$ be a finite complex number, and f be a transcendental meromorphic function with $\overline{N}(r,1/(f^n)^{(k)})=S(r,f)$, then $f(f^n)^{(k)}-a$ has infinitely many zeros.

Proof Let

$$g=(f^n)^{(k)}, \varphi=fg-1 \tag{10}$$

We suppose, to the contrary, that $f(f^n)^{(k)}-a$ has only finitely many zeros, since f is transcendental, then

$$N\left(r,\frac{1}{f(f^n)^{(k)}-a}\right)=S(r,f) \tag{11}$$

By $\varphi=fg-1$, we get $f=(\varphi+1)/g$, then $N\left(r,\frac{1}{f}\right)\leqslant N\left(r,\frac{1}{\varphi+1}\right)+S(r,f)$. Using (10), we have $\frac{1}{f}=\frac{g}{\varphi+1}, \frac{1}{f^2}=\frac{1}{\varphi+1}\cdot\frac{(f^n)^{(k)}}{f}$. Therefore, $m\left(r,\frac{1}{f}\right)\leqslant 2m\left(r,\frac{1}{f}\right)=m\left(r,\frac{1}{f^2}\right)\leqslant m\left(r,\frac{1}{\varphi+1}\right)+S(r,f)$.

Hence

$$T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)=m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+O(1)\leqslant m\left(r,\frac{1}{\varphi+1}\right)+N\left(r,\frac{1}{\varphi+1}\right)+S(r,f)\leqslant$$
$$T\left(r,\frac{1}{\varphi+1}\right)+S(r,f)=T(r,\varphi+1)+S(r,f)=T(r,f(f^n)^{(k)})+S(r,f) \tag{12}$$

On the other hand, by the second fundamental theorem and (11), we have

$$T(r,f(f^n)^{(k)})\leqslant\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f(f^n)^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f(f^n)^{(k)}-a}\right)+S(r,f)\leqslant$$
$$\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f(f^n)^{(k)}}\right)+S(r,f) \tag{13}$$

Let f has a pole z_0 of order p, by $\varphi+1=fg$ and $n\geqslant k+2$, we get z_0 is a pole of $\varphi+1$ of multiplicity $p+(np+k)\geqslant 1+(3+k)=k+4$, thus

$$\overline{N}(r,f)\leqslant\frac{1}{k+4}N(r,\varphi+1)+S(r,f)\leqslant\frac{1}{k+4}T(r,\varphi+1)+S(r,f)=\frac{1}{k+4}T(r,f(f^n)^{(k)})+S(r,f) \tag{14}$$

Let f has a zero z_1 of order q, by $\varphi+1=fg$ and $n\geqslant k+2$, we get z_1 is a zero of $\varphi+1$ of multiplicity $q+(nq-k)\geqslant 1+(k+2-k)=3$. Since $\overline{N}(r,1/(f^n)^{(k)})=S(r,f)$. Thus, we have

$$\overline{N}\left(r,\frac{1}{f(f^n)^{(k)}}\right)\leqslant\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)\leqslant\frac{1}{3}N(r,f(f^n)^{(k)})\leqslant\frac{1}{3}T(r,f(f^n)^{(k)}) \tag{15}$$

According to (13),(14) and (15), we have

$$\frac{2k+5}{3(k+4)}T\left(r,\frac{1}{f(f^n)^{(k)}}\right)\leqslant S(r,f) \tag{16}$$

By(12) and (16), we obtain $T(r,f)\leqslant S(r,f)$. This contradicts the fact that f is transcendental, and hence $f(f^n)^{(k)}-a$has infinitely many zeros.

3 Proof of theorems

Proof of Theorem 1 Without loss of generality, we may assume that $D=\{z\in\mathbf{C}\,|\,|z|<1\}$. Suppose, to the contrary, that $\mathcal{F}$ is not normal in D. Without loss of generality, we assume that $\mathcal{F}$ is not normal at $z_0=0$. Then, by Lemma 1, there exist a sequence $\{z_j\}$ of complex numbers with $z_j\to 0(j\to\infty)$, a sequence $\{f_j\}$ of F; and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j\to 0$, such that

$$g_j(\xi)=\rho_j^{-\frac{k}{n+1}}f_j(z_j+\rho_j\xi) \tag{17}$$

converges uniformly to a non-constantmeromorphic function $g(\xi)$ in $\mathbf{C}$ with respect to the spherical metric. Moreover, $g(\xi)$ is of order at most 2. Hurwitz's theorem implies that $\overline{N}(r,1/(g^n)^{(k)})=S(r,g)$.

By (17), we have

$$f_j(z_j+\rho_j\xi)(f_j^n(z_j+\rho_j\xi))^{(k)}-a=g_j(\xi)(g_j^n(\xi))^{(k)}-a\to g(\xi)(g^n(\xi))^{(k)}-a \tag{18}$$

with respect to the spherical metric.

If $g(g^n)^{(k)}\equiv a$, then g has no zeros. Of course, g also has no poles. Since g is a non-constant meromorphic function of order at most 2, then there exist constants c_i such that $(c_1,c_2)\neq(0,0)$, and

$$g(\xi)=e^{c_0+c_1\xi+c_2\xi^2} \tag{19}$$

Obviously, this is contrary to the case$g(g^n)^{(k)}\equiv a$. Hence $g(g^n)^{(k)}\not\equiv a$.

By Lemma 2 and Lemma 3, the function $g(g^n)^{(k)}-a$ has at least two distinct zeros. Let ξ_0 and ξ_0^* be two distinct zeros of $g(g^n)^{(k)}-a$.

We choose a positive number δ small enough such that $D_1\cap D_2=\varnothing$ and such that $g(g^n)^{(k)}-a$ has no other zeros in $D_1\cup D_2$ except for ξ_0 and ξ_0^*, where

$$D_1=\{\xi\in\mathbf{C}\,|\,|\xi-\xi_0|<\delta\},D_2=\{\xi\in\mathbf{C}\,|\,|\xi-\xi_0^*|<\delta\} \tag{20}$$

By(18) and Hurwitz's theorem, for sufficiently large j there exist points $\xi_j\in D_1$, $\xi_j^*\in D_2$ such that

$$f_j(z_j+\rho_j\xi_j)(f_j^n(z_j+\rho_j\xi_j))^{(k)}-a=0,f_j(z_j+\rho_j\xi_j^*)(f_j^n(z_j+\rho_j\xi_j^*))^{(k)}-a=0$$

By the assumption inTheorem 1, $f(f^n)^{(k)}$ and $g(g^n)^{(k)}$ share a IM. For any integer m, it follows that

$$f_m(z_j+\rho_j\xi_j)(f_m^n(z_j+\rho_j\xi_j))^{(k)}-a=0,f_m(z_j+\rho_j\xi_j^*)(f_m^n(z_j+\rho_j\xi_j^*))^{(k)}-a=0$$

We fix m and note that $z_j+e_j\xi_j\to0,z_j+e_j\xi_j^*\to0,if\ j\to\infty$. we get $f_m(0)(f_m^n(0)^{(k)}-a=0$. Since the zeros of $f_m(z)(f_m^n(z)^{(k)}-a$ have no accumulation points for sufficiently large j, in fact we have $z_j+\rho_j\xi_j=0,z_j+\rho_j\xi_j^*=0$.

Hence $\xi_j=-\dfrac{z_j}{\rho_j},\xi_j^*=-\dfrac{z_j}{\rho_j}$. This contradicts with the facts that $\xi_j\in D_1$, $\xi_j^*\in D_2$, $D_1\cap D_2=\varnothing$.

Theorem 1 is proved completely.

References:

[1] Schiff J.Normal families[M].Berlin:Springer-Verlag,1993.

[2] Hayman W K. Meromorphic functions[M]. Oxford:Clarendon Press,1964.

[3] Yang L. Value distribution theory[M]. Berlin: Springer, 1993.

[4] Yang C C, Yi H X. Uniqueness theory of meromorphic functions[M].Netherlands:Kluwer,2003.

[5] Gu Y X,Pang X C,Fang M L.Theory of Normal families and its application[M]. Beijing:Science Press,2007.

[6] Fang M L,Zalcman L. A note on normality and shared values[J]. J Aust Math Soc,2004,76(1):141-150.

[7] Zeng C P. Normality and shared values with multiple zeros [J]. J Math Anal Appl,2012,394(2):683-686.

[8] Zalcman L. Normal families:new perspectives[J]. Bull Amer Math Soc,1998,35(3):215-230.

[9] Pang X C. Normality conditions for differential polynomials [J]. Chinese Science Bulletin,1988,33(22):1690-1693.

关于分担值的亚纯函数的正规族

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摘要：本文主要研究了关于分担值的亚纯函数的正规性。令F为定义在区域D上的亚纯函数族，k，n（$\geqslant k+2$）为正整数，a为非零复常数。如果对任一对（f，g）$\in F$，都有$f(f^n)^{(k)}$与$g(g^n)^{(k)}$ IM分担a，且$\overline{N}(r,1/(f^n)^{(k)})=S(r,f)$，则$F$在$D$上正规。此结论改进和加强了已有文献中的结论。

关键词：亚纯函数；正规族；分担值

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