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Normal Criteria of Meromorphic Functions Concerning Shared a Holomophic Function*

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Abstract: In this paper, we study the normality criterion concerning shared a holomorphic function. Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $k(\geqslant 1)$, $m(\geqslant 0)$ be two integer, and let $\omega\not\equiv 0$ be a holomorphic function with zeros of multiplicity m in D. If, for any $f\in\mathcal{F}$, the multiplicity of all zeros and poles of f is at least $\max\{m+k, m+1+k/2\}$, and if $ff^{(k)}, gg^{(k)}$ share ω IM for every pair of functions $f,g\in\mathcal{F}$, then $\mathcal{F}$ is normal in D.

Key words: meromorphic functions; normal families; holomorphic function
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1 Introduction and main result

Let D be a domain in $\mathbf{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_n\}\subset\mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞[1-3].

Let $g(z)$ be a meromorphic function, a be a finite complex number. If $f(z)$ and $g(z)$ assume the same zeros, then we say that share a IM (ignoring multiplicity)[1,4-5].

In 2004, Fang and Zalcman[6] got the following results.

Theorem A Suppose that k is a positive integer and $a\neq 0$ is a finite complex number. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. If for each pair of functions $f,g\in\mathcal{F}$, f and g share 0, $f^{(k)}$ and $g^{(k)}$ share a IM in D, and the zeros of f are of multiplicity $\geqslant k+2$, then $\mathcal{F}$ is normal in D.

In 2011, Meng[7] proved the following result.

Theorem B Take a positive integer k and a non-zero complex number a. Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D\subset\mathbf{C}$ such that each $f\in\mathcal{F}$ has only zeros of multiplicity at least $k+1$. For each pair $(f,g)\in\mathcal{F}$ if $ff^{(k)}$ and $gg^{(k)}$ share a IM, then $\mathcal{F}$ is normal in D.

Theorem C Take a positive integer $k\geqslant 2$ and a non-zero complex number a. Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D\subset\mathbf{C}$ such that each $f\in\mathcal{F}$ has only zeros of multiplicity at least k. For each pair $(f,g)\in\mathcal{F}$, if $ff^{(k)}$ and $gg^{(k)}$ share a IM, then $\mathcal{F}$ is normal in D.

In 2012, Zeng[8] proved the following result.

Theorem D Let k be a positive integer, $a(\neq 0)$ and b be two finite values. Let $\mathcal{F}$ be a family of meromor-

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phic functions defined in D, all of whose zeros have multiplicity at least k and $f^{(k)}(z)=b$ when $f(z)=0$. If for each pair of functions f and g in $\mathcal{F}$, $ff^{(k)}$ and $gg^{(k)}$ share a, then $\mathcal{F}$ is normal in D.

It is natural to ask whether there exist normality theorems corresponding to Theorem B if a is a function? In this paper, we study the problem and obtain the following theorem.

Theorem 1 Let $\mathcal{F}$ be a family of meromorphic functions dened in D, let $k(\geqslant 1)$, $m(\geqslant 0)$ be two integer, and let $\omega\not\equiv 0$ be a holomorphic function with zeros of multiplicity m in D. If, for any $f\in\mathcal{F}$, the multiplicity of all zeros and poles of f is at least $\max\{m+k, m+1+k/2\}$, and if $ff^{(k)}$, $gg^{(k)}$ share ω IM for every pair of functions $f,g\in\mathcal{F}$, then $\mathcal{F}$ is normal in D.

Corollary 1 Let $\mathcal{F}$ be a family of holomorphic functions defined in D, let $k(\geqslant 1)$, $m(\geqslant 0)$ be two integer, and let $\omega\not\equiv 0$ be a holomorphic function with zeros of multiplicity m in D. If, for any $f\in\mathcal{F}$, f has only zeros of multiplicity $m+k$ at least, and if $ff^{(k)}$, $gg^{(k)}$ share ω IM for every pair of functions $f,g\in\mathcal{F}$, then $\mathcal{F}$ is normal in D.

Corollary 2 Let $\mathcal{F}$ be a family of holomorphic functions defined in D, let $k(\geqslant 1)$, $m(\geqslant 0)$ be two integer, and let $\omega\not\equiv 0$ be a holomorphic function with zeros of multiplicity m in D. If, for any $f\in\mathcal{F}$, f has only zeros of multiplicity $m+k$ at least, and $ff^{(k)}\neq\omega(z)$, then $\mathcal{F}$ is normal in D.

Corollary 3 Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $k(\geqslant 1)$, $m(\geqslant 0)$ be two integer, and let $\omega\not\equiv 0$ be a holomorphic function with zeros of multiplicity m in D. If, for any $f\in\mathcal{F}$, the multiplicity of all zeros and poles of f is at least $\max\{m+k, m+1+k/2\}$, and $ff^{(k)}\neq\omega(z)$, then $\mathcal{F}$ is normal in D.

Remark Clearly, when $m=0$, Theorem 1 and Corollary 1 extends Theorem B and C.

2 Some lemmas

Lemma 1(Zalcman's Lemma)[9-10] Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ and α be a real number satisfying $-1<\alpha<1$. Then if $\mathcal{F}$ is not normal at a point $z_0\in\Delta$, there exist, for each $-1<\alpha<1$: 1) a real number r, $r<1$; 2) points z_n, $|z_n|<r$; 3) positive numbers $\rho_n, \rho_n\to 0^+$; 4) functions f_n, $f_n\in\mathcal{F}$, such that $g_n(\xi)=\dfrac{f_n(z_n+\rho_n\xi)}{\rho_n^{\alpha}}$, spherically uniformly on compact subsets of $\mathbf{C}$, where $g(\xi)$ is a non-constant meromorphic function and $g^{\#}(\xi)\leqslant g^{\#}(0)=1$. Moreover, the order of g is not greater than 2.

By using the method of Lemma 2.6 in Meng[11], we have the following Lemmas.

Lemma 2 Let f be a transcendental meromorphic function whose zeros have multiplicity at least k, and let $p(z)(\not\equiv 0)$ be a polynomial. Then $ff^{(k)}-p(z)$ has infinitely many zeros.

Lemma 3 Let $p(z)=a_m z^m+a_{m-1}z^{m-1}+\cdots+a_1 z+a_0$ be a polynomial, where $a_m(\neq 0), a_{m-1},\cdots,a_0$ are constants. Let $k\geqslant 1$ be an integer, if f be a nonconstant polynomial, and all the zeros of f have multiplicity $m+k$ at least, then $ff^{(k)}-p(z)$ has at least two distinct zeros.

Proof Since f is a non-constant polynomial with zeros of multiplicity $m+k$ at least, thus $\deg(ff^{(k)})>\deg p(z)$, then $ff^{(k)}-p(z)$ has at least one zero.

If $ff^{(k)}-p(z)$ has only one zero z_0. We may assume $ff^{(k)}-p(z)=A(z-z_0)^l$, where A is a non-zero constant, l is a positive integer. Then, $l=\deg(ff^{(k)})>m+1$, therefore,

$$(ff^{(k)})^{(m)}-Al(l-1)\cdots(l-m+1)(z-z_0)^{l-m}=a_m\cdot m!\neq 0,$$

$$(ff^{(k)})^{(m+1)}-Al(l-1)\cdots(l-m)(z-z_0)^{l-m-1}=0,$$

thus z_0 is the unique zero of $(ff^{(k)})^{(m+1)}$. Since f is a non-constant polynomial with zeros of multiplicity $m+k$ at least, then z_0 is a zero of f, thus $(ff^{(k)})^{(m)}(z_0)=0$, it contradicts with $(ff^{(k)})^{(m)}(z_0)=a_m\cdot m!\neq 0$. Thus, $ff^{(k)}-p(z)$ has at least two distinct zeros.

Lemma 4 Let f be a nonconstant rational function whose zeros and poles have multiplicity at least $m+1+k/2$, k be a positive integer and $p(z)$ be a non-zero polynomial of degree m, then $ff^{(k)}-p(z)$ has at least two distinct zeros.

Proof Let

$$f=A\frac{(z-\alpha_1)^{m_1}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}\cdots(z-\beta_t)^{n_t}}. \tag{1}$$

Where A is a non-zero constant and $m_i\geqslant m+1+k/2(i=1,2,\cdots,s)$, $n_j\geqslant m+1+k/2(i=1,2,\cdots,t)$.

Moreover, we denote

$$m_1+m_2+\cdots+m_s=M\geqslant(m+1+k/2)s, n_1+n_2+\cdots+n_t=N\geqslant(m+1+k/2)t. \tag{2}$$

By (1), we obtain

$$f^{(k)}(z)=A\frac{(z-\alpha_1)^{m_1-k}\cdots(z-\alpha_s)^{m_s-k}g(z)}{(z-\beta_1)^{n_1+k}\cdots(z-\beta_t)^{n_t+k}}, \tag{3}$$

where $g(z)$ is a polynomial of degree at most $k(s+t-1)$. Thus (1) together with (3) imply

$$ff^{(k)}(z)=A^2\frac{(z-\alpha_1)^{2m_1-k}\cdots(z-\alpha_s)^{2m_s-k}g(z)}{(z-\beta_1)^{2n_1+k}\cdots(z-\beta_t)^{2n_t+k}}, \tag{4}$$

Differentiate both sides of (4), we obtain

$$(ff^{(k)}(z))'=\frac{(z-\alpha_1)^{2m_1-k-1}\cdots(z-\alpha_s)^{2m_s-k-1}g_1(z)}{(z-\beta_1)^{2n_1+k+1}\cdots(z-\beta_t)^{2n_t+k+1}}, \tag{5}$$

where $g_1(z)$ is a polynomial of degree at most $(k+1)(s+t-1)$.

$$(ff^{(k)}(z))^{(m+1)}=\frac{(z-\alpha_1)^{2m_1-k-m-1}\cdots(z-\alpha_s)^{2m_s-k-m-1}g_{m+1}(z)}{(z-\beta_1)^{2n_1+k+m+1}\cdots(z-\beta_t)^{2n_t+k+m+1}}, \tag{6}$$

where $g_{m+1}(z)$ is a polynomial of degree at most $(k+m+1)(s+t-1)$.

Next, we discuss two cases.

Case 1 If $ff^{(k)}-p(z)$ has a unique zero z_0, then let

$$ff^{(k)}(z)-p(z)=\frac{B(z-z_0)^l}{(z-\beta_1)^{2n_1+k}\cdots(z-\beta_t)^{2n_t+k}}, \tag{7}$$

where B is a non-zero constant, l is a positive integer.

Here, we discuss two subcases.

Case 1.1 Suppose $m\geqslant l$. Differentiate both sides of (7), we have

$$(ff^{(k)}(z))^{(m+1)}-p^{(m+1)}(z)=\frac{CQ_{m+1}(z)}{(z-\beta_1)^{2n_1+k+m+1}\cdots(z-\beta_t)^{2n_t+k+m+1}}, \tag{8}$$

where $Q_{m+1}(z)=(l-2N-tk)(l-2N-tk-1)\cdots(l-2N-tk-m)z^{(m+1)t-(m-l+1)}+\cdots+b_1z+b_0$ is a polynomial, b_1, b_0 are constants.

Comparing (4) with (7), by $m\geqslant l$, we have $2N+tk+m=\deg g+2M-sk\leqslant k(s+t-1)+2M-sk$, hence, $2(M-N)\geqslant m+k>0$. From (6) and (8), we get $2M-s(k+m+1)\leqslant(m+1)t-(m-l+1)$. Hence

$$m-l+1\leqslant(m+1)t+s(k+m+1)-2M\leqslant\frac{(m+1)N}{m+1+k/2}+\frac{(k+m+1)M}{m+1+k/2}-2M<$$

$$\frac{(m+1)M}{m+1+k/2}+\frac{(k+m+1)M}{m+1+k/2}-2M=0.$$

i. e., $l-m>1$, it contradicts with $m\geqslant l$.

Case 1.2 Suppose $m<l$. Differentiate both sides of (7), we have

$$(ff^{(k)}(z))^{(m+1)}-p^{(m+1)}(z)=\frac{(z-z_0)^{l-m-1}R(z)}{(z-\beta_1)^{2n_1+k+m+1}\cdots(z-\beta_t)^{2n_t+k+m+1}}, \tag{9}$$

where $R(z)=B(l-2N-tk)(l-2N-tk-1)\cdots(l-2N-tk-m)z^{(m+1)t}+\cdots+c_1z+c_0$ is a polynomial, c_1, c_0 are constants.

Next, we discuss three subcases.

Case 1.2.1 If $l<2N+tk+m$. By (4), (7), similar to the proof of Subcase 1.1, we get $M>N$. From (6), (9), we have $2M-s(k+m+1)\leqslant(m+1)t$. i. e.,

$$2M\leqslant(m+1)t+s(k+m+1)\leqslant\frac{(m+1)N}{m+1+k/2}+\frac{(k+m+1)M}{m+1+k/2}<\frac{(m+1)M}{m+1+k/2}+\frac{(k+m+1)M}{m+1+k/2}=2M,$$

which is impossible.

Case 1.2.2 If $l=2N+tk+m$. Suppose $M>N$, by (6),(9), we get $2M-s(k+m+1)\leqslant(m+1)t$. i.e.,

$$2M\leqslant(m+1)t+s(k+m+1)\leqslant\frac{(m+1)N}{m+1+k/2}+\frac{(k+m+1)M}{m+1+k/2}<\frac{(m+1)M}{m+1+k/2}+\frac{(k+m+1)M}{m+1+k/2}=2M,$$

which is contradiction.

Thus, $M\leqslant N$. By (6), (9), we obtain $l-m-a\leqslant(k+m+1)(s+t-1)$. Since $l=2N+tk+m$, then $2N-1+tk=l-m-1\leqslant(k+m+1)(s+t-1)$, i.e.,

$$2N\leqslant(k+m+1)(s+t-1)+1-tk<s(k+m+1)+t(m+1)\leqslant\frac{M(k+m+1)}{m+1+k/2}+\frac{N(m+1)}{m+1+k/2}\leqslant 2N,$$

which is contradiction.

Case 1.2.3 If $l>2N+tk+m$. Suppose $M\leqslant N$, by (4), (7), we have $l\leqslant 2N+tk+m$, which is a contradiction. Thus, $M>N$. From (6), (9), we have $2M-s(k+m+1)\leqslant t(m+1)$. Then,

$$2M\leqslant s(k+m+1)+t(m+1)\leqslant\frac{M(k+m+1)}{m+1+k/2}+\frac{N(m+1)}{m+1+k/2}<2M,$$

which is impossible.

Case 2 If $ff^{(k)}-p(z)$ has no zero, then $l=0$ for (7). Proceeding as in the proof for case 1, we also obtain a contradiction.

Therefore, Lemma 4 is proved complectly.

3 Proof of theorem

3.1 Proof of Theorem 1

For any point $z_0\in D$, either $\omega(z_0)=0$ or $\omega(z_0)\neq 0$.

Next, we consider the following two cases.

Case 1 If $\omega(z_0)=0$. We may assume $z_0=0$. Then, $\omega(z)=z^n h(z)$ where $n\geqslant 1$ is a positive integer and $h(z)$ is holomorphic in D with $h(0)=1$ for D. Let $\mathcal{F}_1=\left\{F_j=\frac{f_j(z)}{z^n}, f_j\in\mathcal{F}\right\}$. We shall prove $\mathcal{F}_1$ is normal at origin. Suppose not, by Lemma 1, there exist points $z_i\to 0$, positive numbers $\rho_i\to 0$ and $F_i\in\mathcal{F}_1$ such that $g_i(\xi)=\rho_i^{\frac{-k}{2}}F_i(z_i+\rho_i\xi)$ converges uniformly to a non-constant meromorphic function $g(\xi)$ in $\mathbf{C}$ with respect to the spherical metric. Moreover, $g(\xi)$ is of order at most 2.

Now, we distinguish two cases.

Case 1.1 There exists a subsequence of $\frac{z_i}{\rho_i}$, we may still denote it as $\frac{z_i}{\rho_i}$ such that $\frac{z_i}{\rho_i}\to\alpha,\alpha$ is a finite complex number. We have

$$G_i(\xi):=\frac{f_i(\rho_i\xi)}{\rho_i^{n+\frac{k}{2}}}=\frac{(\rho_i\xi)^n F_i\left(z_i+\rho_i\left(\xi-\frac{z_i}{\rho_i}\right)\right)}{(\rho_i)^n\ (\rho_i)^{\frac{k}{2}}}\to\xi^n g(\xi-\alpha)=\widetilde{g}(\xi),$$

spherically locally uniformly in $\mathbf{C}$. Hence,

$$G_i(\xi)G_i^{(k)}(\xi)-\frac{\omega(\rho_i\xi)}{\rho_i^n}=\frac{f_i(\rho_i\xi)f_i^{(k)}(\rho_i\xi)-\omega(\rho_i\xi)}{\rho_i^n}\to\widetilde{g}(\xi)\widetilde{g}^{(k)}(\xi)-\xi^n$$

spherically locally uniformly in $\mathbf{C}$.

For any $f\in\mathcal{F}$, he multiplicity of every zero of f is $\max\{m+k,m+1+k/2\}$ at least, the multiplicity of every zero of $\widetilde{g}$ is $\max\{m+k,m+1+k/2\}$ at least, from Lemma 2～4, we get $\widetilde{g}(\xi)\widetilde{g}^{(k)}(\xi)-\xi^n\not\equiv 0$, and $\widetilde{g}(\xi)\widetilde{g}^{(k)}(\xi)-\xi^n$ has two distinct zeros at least.

Let ξ_0 and ξ_0^* be two distinct zeros of $\widetilde{g}(\xi)\widetilde{g}^{(k)}(\xi)-\xi^n$.

We choose a positive number δ small enough such that $D_1\cap D_2=\varnothing$ and such that $\widetilde{g}(\xi)\widetilde{g}^{(k)}(\xi)-\xi^n$ has no other zeros in $D_1\cup D_2$ except for ξ_0 and ξ_0^*, where

$$D_1=\left\{\xi\in\mathbf{C}\,\Big|\,|\xi-\xi_0|<\delta\right\},D_2=\left\{\xi\in C\,\Big|\,|\xi-\xi_0^*|<\delta\right\}. \tag{10}$$

By Hurwitz's theorem, for sufficiently large i there exist points $\xi_i\in D_1,\xi_i^*\in D_2$ such that

$$f_i(\rho_i\xi_i)f_i^{(k)}(\rho_i\xi_i)-\omega(\rho_i\xi_i)=0,f_i(\rho_i\xi_i^*)f_i^{(k)}(\rho_i\xi_i^*)-\omega(\rho_i\xi_i^*)=0.$$

By the assumption in Theorem 1, $ff^{(k)}$ and $gg^{(k)}$ share ω IM for every pair of functions $f,g\in\mathcal{F}$. Then, for any integer r, it follows that

$$f_r(\rho_i\xi_i)f_r^{(k)}(\rho_i\xi_i)-\omega(\rho_i\xi_i)=0,f_r(\rho_i\xi_i^*)f_r^{(k)}(\rho_i\xi_i^*)-\omega(\rho_i\xi_i^*)=0.$$

We fix r and note that $\rho_i\xi_i\to0,\rho_i\xi_i^*\to0$ if $i\to\infty$. We get $f_r(0)f_r^{(k)}(0)-\omega(0)=0$.

Since the zero of $f_r(z)f_r^{(k)}(z)-\omega(z)$ have no accumulation points for sufficiently large i, in fact we have $\rho_i\xi_i=\rho_i\xi_i^*=0$. Hence $\xi_i=\xi_i^*=0$. This contradicts with the facts that $\xi_i\in D_1,\xi_i^*\in D_2,D_1\cap D_2=\varnothing$. Therefore, $\mathcal{F}_1$ is normal at 0.

Case 1.2 There exists a subsequence of $\frac{z_i}{\rho_i}$, we may still denote it as $\frac{z_i}{\rho_i}$ such that $\frac{z_i}{\rho_i}\to\infty$. Then,

$$F_i^{(k)}=\begin{cases}f_i^{(k)}/z^n+\sum_{t=0}^{k-1}C_{k-t}(F_i^{(t)}(z)/z^{k-t}),\text{if } n>k,\\ f_i^{(k)}/z^n+\sum_{t=0}^{k-n}C_{k-t}(F_i^{(t)}(z)/z^{k-t}),\text{if } n\leqslant k,\end{cases} \tag{11}$$

where $C_l,l=i,2,\cdots,k$ are constant. Thus,

$$\rho_i^{-\frac{k}{2}}g_i^{(k)}(\xi)=F_i^{(k)}(z_i+\rho_i\xi)=\frac{f_i^{(k)}(z_i+\rho_i\xi)}{(z_i+\rho_i\xi)^n}+\frac{C_1}{z_i+\rho_i\xi}F_i^{(k-1)}(z_i+\rho_i\xi)+\cdots+\frac{C_k}{(z_i+\rho_i\xi)^k}F_i(z_i+\rho_i\xi) \tag{12}$$

where $C_l=0,l=n+1,\cdots,k$, if $n<k$. Obviously,

$$\lim_{i\to\infty}\frac{1}{h(z_i+\rho_i\xi)}=1 \tag{13}$$

uniformly on compact subset of $\mathbf{C}$. At the same time,

$$\lim_{i\to\infty}\frac{C_l\rho_n^l}{(z_i+\rho_i\xi)^l}=\lim_{i\to\infty}\frac{C_l}{(z_i/\rho_i+\xi)^l}=0,l=1,2,\cdots,k \tag{14}$$

uniformly on compact subset of $\mathbf{C}$. By (12),(13) and (14), we have

$$\frac{f_i^{(k)}(z_i+\rho_i\xi)}{\omega(z_i+\rho_i\xi)}=\frac{f_i^{(k)}(z_i+\rho_i\xi)}{(z_i+\rho_i\xi)^nh(z_i+\rho_i\xi)}=\frac{1}{h(z_i+\rho_i\xi)}\left[\rho_i^{-\frac{k}{2}}g_i^{(k)}(\xi)-\frac{C_l\rho_i^{1-\frac{k}{2}}g_i^{(k-1)}(\xi)}{z_i+\rho_i\xi}-\cdots-\frac{C_l\rho_i^{\frac{k}{2}}g_i(\xi)}{(z_i+\rho_i\xi)^k}\right] \tag{15}$$

Thus,

$$\frac{f_i(z_i+\rho_i\xi)f_i^{(k)}(z_i+\rho_i\xi)}{\omega(z_i+\rho_i\xi)}-1\to g(\xi)g^{(k)}(\xi)-1 \tag{16}$$

spherically locally uniformly in $\mathbf{C}-\{\xi\,|\,g(\xi)=\infty\}$.

If $gg^{(k)}\equiv1$, then g has no zeros. Of course, g also has no poles. Since g is a non-constant meromorphic function of order at most 2, then there exist constants c_i such that $(c_1,c_2)\neq(0,0)$, and $g(\xi)=e^{c_0+c_1\xi+c_2\xi^2}$. Obviously, this is contrary to the case $gg^{(k)}\equiv1$. Hence $gg^{(k)}\not\equiv1$. From Lemma 2, Lemma 3 and Lemma 4, we get $g(\xi)g^{(k)}(\xi)-1$ has two distinct zeros at least.

Let ξ_1 and ξ_1^* be two distinct zeros of $g(\xi)g^{(k)}(\xi)-1$.

We choose a positive number σ small enough such that $D_1\cap D_2=\varnothing$ and such that $g(\xi)g^{(k)}(\xi)-1$ has no other zeros in $D_1\cup D_2$ except for ξ_1 and ξ_1^*, where

$$D_1=\{\xi\in\mathbf{C}\,|\,|\xi-\xi_1|<\sigma\},D_2=\{\xi\in\mathbf{C}\,|\,|\xi-\xi_1^*|<\sigma\} \tag{17}$$

By Hurwitz's theorem, for sufficiently large i there exist points $\zeta_i\in D_1,\zeta_i^*\in D_2$ such that
Similar to the proof of Case 1.1, we get a contradiction. Therefore, $\mathcal{F}_1$ is normal at 0.

It remains to prove that $\mathcal{F}$ is normal at origin. Suppose f_{jk} be a sequence of functions in $\mathcal{F}$. Since $\mathcal{F}_1$ is normal at 0, there exists $\Delta_r=\{z:|z|<r\}$, $\mathcal{F}_1$ is normal on Δ_r, then there exist $\delta<\frac{r}{2}$ such that F_{jk} uniformly converges to a meromorphic function $u(z)$ or ∞ on $\Delta_{2\delta}$. Noting $F_{jk}(0)=\infty$, we deduce that exists positive constant R such that $|F_{jk}|\geqslant R$ for all $z\in\Delta_\delta$. Thus, $f_{jk}\neq 0$ for all $z\in\Delta_\delta$ and for all k. Therefore, $1/f_{jk}$ is analytic in Δ_δ. Therefore, we have

$$\left|\frac{1}{f_{jk}(z)}\right|=\left|\frac{1}{F_{jk}(z)}\ \frac{1}{|z|^n}\right|\leqslant R\,\frac{2^n}{\delta^n},\ |z|=\frac{\delta}{2}. \tag{18}$$

By Montel's Theorem, $\mathcal{F}$ is normal at $z=0$.

Case 2 If $\omega(z_0)\neq 0$. Suppose that $\mathcal{F}$ is not normal at z_0. By Lemma 1, there exist points $z_n\to z_0$, $\rho_n\to 0$, $f_n\to\mathcal{F}$ such that $\rho_n^{-\frac{k}{2}}f_n(z_n+\rho_n\xi)\to g(\xi)$ spherically locally uniformly in $\mathbf{C}$, $g(\xi)$ is a nonconstant meromorphic function in $\mathbf{C}$, and $g^\#(\xi)\leqslant 1$. Moreover, the order of $g(\xi)$ is not greater than 2.

Since for any $f\in\mathcal{F}$, the multiplicity of zeros of f is $\max\{m+k, m+1+k/2\}$ at least, then the multiplicity of zeros of g is $\max\{m+k, m+1+k/2\}$ at least.

Thus, from Lemma 2, Lemma 3 and Lemma 4, we get $gg^{(k)}-\omega(z_0)\not\equiv 0$, and $gg^{(k)}-\omega(z_0)$ has two distinct zeros at least. Similar to the proof of Case 1.1, we get a contradiction. Thus, $\mathcal{F}$ is normal at z_0. Theorem 1 is proved completely.

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亚纯函数分担全纯函数的正规族

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摘要:主要研究了亚纯函数分担全纯函数的正规族问题,证明了:如果$\mathcal{F}$是区域D上的亚纯函数,$k(\geqslant 1)$, $m(\geqslant 0)$为两个整数,$\omega\not\equiv 0$为一个全纯函数,在D内其零点的重级为m。如果对任意的$f\in\mathcal{F}$,f的所有零点及极点的重级至少为$\max\{m+k, m+1+k/2\}$,且对任意的$f,g\in\mathcal{F}$都有$ff^{(k)}$,$gg^{(k)}$ IM 分担ω,则$\mathcal{F}$在D正规。

关键词:亚纯函数;正规族;全纯函数

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