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解双边空间分数阶对流扩散方程的二阶隐式有限差分法*

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摘要:给出一类解变系数双边空间分数阶对流扩散方程的隐式有限差分格式，并证明这类格式当分数阶导数 $\alpha\in[\sqrt{17}-1/2,2]$ 时无条件稳定且由此得出收敛阶为 $O(\Delta t+h^2)$。最后给出数值算例验证。

关键词:变系数双边空间分数阶对流扩散偏微分方程;有限差分格式;无条件稳定;收敛阶

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近几十年来，分数阶偏微分方程在工程、物理、金融、流体等领域得到了越来越广泛的应用[1-3]。

国内外学者在分数阶偏微分方程的有限差分解法中做了很多工作[4-9]。本文研究下列变系数双边空间分数阶偏微分方程的初边值问题：

$$\frac{\partial u(x,t)}{\partial t}=-p(x)\frac{\partial u(x,t)}{\partial x}+c_+(x)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}+c_-(x)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}+s(x,t),\tag{1}$$

$$x\in[L,\mathbf{R}],t\in[0,T],\tag{2}$$

$$u(L,t)=u(\mathbf{R},t)=0,t\in[0,T]。\tag{3}$$

$$u(x,0)=\varphi(x),x\in[L,\mathbf{R}]。\tag{4}$$

本文将利用经典 Grünwald-Letnikov 算子和移位 Grünwald-Letnikov 算子[5]进行加权平均构造新的算子先来近似求解方程(1)中左、右导数项，对流项的离散采用经典的向后迎风格式并且保留其截断误差项中的二阶导数项，并对其$\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}$，$\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}$采用第 n 时刻的中心差分做为修正，从而给出空间上具有二阶精度的隐式有限差分格式，并且用能量不等式方法证明其稳定性及收敛性。

1 预备知识

在方程(1)中，关于左、右 α 阶导数的 Riemann-Liouville 形式为：

$$\begin{cases}\dfrac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}=\dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dx^n}\displaystyle\int_L^x\dfrac{f(\xi)}{(x-\xi)^{\alpha+1-n}}\mathrm{d}\xi,\\ \dfrac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}=\dfrac{(-1)^n}{\Gamma(n-\alpha)}\dfrac{d^n}{dx^n}\displaystyle\int_R^x\dfrac{f(\xi)}{(\xi-x)^{\alpha+1-n}}\mathrm{d}\xi,\end{cases}\tag{5}$$

其中，$n>\alpha>n-1\geqslant0$(n 为整数)，Γ 是伽马函数。

计算左、右 Riemann-Liouville 导数的经典 Grünwald 公式和移位 Grnüwald 公式分别为：

$$\begin{cases}\dfrac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}=\lim\limits_{h\to0}\dfrac{1}{h^\alpha}\displaystyle\sum_{k=0}^{K_+}g_k u(x-kh,t)+O(h),\\ \dfrac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}=\lim\limits_{h\to0}\dfrac{1}{h^\alpha}\displaystyle\sum_{k=0}^{K_+}g_k u(x-(k-1)h,t)+O(h),\end{cases}\tag{6}$$

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$$\begin{cases}\dfrac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}=\lim\limits_{h\to 0}\dfrac{1}{h^\alpha}\sum\limits_{k=0}^{K_-} g_k u(x+kh,t)+O(h),\\ \dfrac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}=\lim\limits_{h\to 0}\dfrac{1}{h^\alpha}\sum\limits_{k=0}^{K_-} g_k u(x+(k-1)h,t)+O(h),\end{cases}\tag{7}$$

其中,K_+,K_-是正整数,g_k 是 Grünwald 系数,且定义如下:

$$g_0=1,g_k=(-1)^k\binom{\alpha}{k}=(-1)^k\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}。\tag{8}$$

另外,g_k 还具有如下性质:

$$\begin{cases}g_1=-\alpha<0,1\geqslant g_2\geqslant g_3\geqslant\cdots\geqslant 0,\\ \sum\limits_{k=0}^{\infty} g_k=0,\sum\limits_{k=0}^{m} g_k\leqslant 0(m\geqslant 1)。\end{cases}\tag{9}$$

2 二阶隐式差分格式

为了给出数值计算公式,用值 v_i^n 表示函数 $u(x,t)$在点(x_i,t_n)的数值解,类似地设:

$$t_n=n\Delta t,x_i=L+ih,p_i=p(x_i),c_{+,i}=c_+(x_i),c_{-,i}=c_-(x_i),s_i^n=s(x_i,t_n),$$

其中,$i=0,1,\cdots,K$,$n=0,1,\cdots,\leqslant\dfrac{T}{\tau}$,$\tau$ 和 $h=\dfrac{R-L}{K}$分别表示时间和空间上的步长。

将(6)式和(7)式各自加权平均近似计算$\dfrac{\partial^\alpha u(x_i,t_{n+1})}{\partial_+ x^\alpha}$,$\dfrac{\partial^\alpha u(x_i,t_{n+1})}{\partial_- x^\alpha}$如下:

$$\begin{aligned}\frac{\partial^\alpha u(x_i,t_{n+1})}{\partial_+ x^\alpha}&\approx\frac{1}{h^\alpha}\Big[(1-\varepsilon)\sum_{k=0}^{i} g_k v_{i-k}^{n+1}+\varepsilon\sum_{k=0}^{i+1} g_k v_{i-k+1}^{n+1}\Big],\\ \frac{\partial^\alpha u(x_i,t_{n+1})}{\partial_- x^\alpha}&\approx\frac{1}{h^\alpha}\Big[(1-\varepsilon)\sum_{k=0}^{K-i} g_k v_{i+k}^{n+1}+\varepsilon\sum_{k=0}^{K-i+1} g_k v_{i+k-1}^{n+1}\Big],\end{aligned}\tag{10}$$

其中,ε 是加权系数且 $0<\varepsilon<1$。

下面证明当 $\varepsilon=\dfrac{\alpha}{2}$时,上述差分近似(10)具有二阶精度。

定理 1 设 $f\in L_1(\mathbf{R})$且 $f\in C^{\alpha+1}(\mathbf{R})$,设:

$$B_h f(x)=\frac{(1-\varepsilon)}{h^\alpha}\sum_{k=0}^{\infty} g_k f(x-kh)+\frac{\varepsilon}{h^\alpha}\sum_{k=0}^{\infty} g_k f(x-(k-1)h),\tag{11}$$

$$Bf(x)=\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}_+ x^\alpha}\ (Bf(x)\text{是 Liouville 分数阶导数定义,积分下限是}-\infty),\tag{12}$$

则当 $\varepsilon=\dfrac{\alpha}{2}$时,对 $\forall x\in\mathbf{R}^1$ 都有 $B_h f(x)=Bf(x)+O(h^2)$成立。

证明 设 $F[f](k)=\overline{f}(k)=\int \mathrm{e}^{ikx}f(x)\mathrm{d}x$ 是 $f(x)$的傅里叶展开,则 $\mathrm{e}^{ikh}\overline{f}(k)$是 $f(x-h)$的傅里叶展开。令

$$A_h f(x)=h^{-\alpha}\sum_{k=0}^{\infty} g_k f(x-(k-p)h)。\tag{13}$$

将上式进行傅里叶展开,并且由(7)式可得:

$$\begin{aligned}F[A_h f](k)&=\frac{1}{h^\alpha}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}\mathrm{e}^{ik(m-p)h}\overline{f}(k)=\frac{1}{h^\alpha}\mathrm{e}^{-ikph}(1-\mathrm{e}^{ikh})^\alpha\,\overline{f}(k)=\\ &\frac{1}{h^\alpha}(-ikh)^\alpha\left(\frac{1-\mathrm{e}^{ikh}}{-ikh}\right)^\alpha\mathrm{e}^{-ikph}\overline{f}(k)=(-ik)^\alpha w(-ikh)\overline{f}(k),\end{aligned}\tag{14}$$

其中,$(iu)^\alpha=\mathrm{sign}\,(u)|u|^\alpha\exp(i\pi\alpha/2)$,$u\in\mathbf{R}$,且

$$w(z)=\left(\frac{1-\mathrm{e}^{-z}}{z}\right)^\alpha\mathrm{e}^{zp}=1+\left(p-\frac{\alpha}{2}\right)z+O(|z|^3),$$

$$w_0(z)=1-\frac{\alpha}{2}z+O(|z|^2)。\tag{15}$$

令 $p=0,1$ 得:

$$w_1(z)=1+\left(1-\frac{\alpha}{2}\right)z+O(|z|^2)。\tag{16}$$

由上式知

$$\overline{w}(z)=(1-\varepsilon)w_0(z)+\varepsilon w_1(z)=(1-\varepsilon)\left(1-\frac{\alpha}{2}z\right)+\varepsilon\left[1+\left(1-\frac{\alpha}{2}\right)z\right]+O(|z|^2)=$$
$$1-\frac{\alpha}{2}(1-\varepsilon)z+\varepsilon\left(1-\frac{\alpha}{2}\right)z+O(|z|^2),\tag{17}$$

则令 $\varepsilon=\frac{\alpha}{2}$,则得 $\overline{w}(z)=1+(O\ |z|^2)$,即对于 $\forall x\in\mathbf{R}$, $|\overline{w}(-ix)-1|\leqslant C\ |x|^2$。

又因为

$$F[B_hf](k)=(-ik)^\alpha[(1-\varepsilon)w_0(-ikh)+\varepsilon w_1(-ikh)]\overline{f}(k)=(-ik)^\alpha\ \overline{f}(k)+$$
$$(-ik)^\alpha(\overline{w}(-ikh)-1)\overline{f}(k)\ =F[Af](k)+\overline{\varphi}(h,k),\tag{18}$$

其中,$\overline{\varphi}(h,k)=(-ik)^\alpha(\overline{w}(-ikh)-1)\overline{f}(k)$ 且 $\varphi(h,k)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{e}^{-ikx}\ \overline{\varphi}(h,k)\mathrm{d}k$。

因为 $f\in L_1(\mathbf{R})$ 且 $f\in C^{\alpha+1}(\mathbf{R})$,则得 $I=\int_{-\infty}^{\infty}(1+|k|)^{\alpha+1}\ |\overline{f}(k)|\ \mathrm{d}k<\infty$。所以 $|\overline{\varphi}(h,k)|\leqslant |k|^\alpha C|kh^2||\overline{f}(k)|$,从而 $|\varphi(h,k)|\leqslant Ch^2$。

因此,对于 $\forall x\in\mathbf{R}$, $B_hf(x)=Bf(x)+O(h^2)$成立。　　证毕

对于右导数项,用同样的证明方法可以得到相似的结论。

定理 2　设 $f\in L_1(\mathbf{R})$且 $f\in C^{\alpha+1}(\mathbf{R})$,设:

$$D_hf(x)=\frac{(1-\varepsilon)}{h^\alpha}\sum_{k=0}^{\infty}g_kf(x+kh)+\frac{\varepsilon}{h^\alpha}\sum_{k=0}^{\infty}g_kf(x+(k-1)h),$$
$$Df(x)=\frac{d^\alpha f(x)}{d_-x^\alpha}(Bf(x)\text{是 Liouville 分数阶导数定义,积分下限是}-\infty),$$

则当 $\varepsilon=\frac{\alpha}{2}$时,对 $\forall x\in\mathbf{R}^1$ 都有 $D_hf(x)=Df(x)+O(h^2)$成立。

注　方程(1)满足边界条件(3),(4),(5)时,Riemann-Liouville 分数阶导数定义和 Liouville 分数阶导数定义是等价的[5]。

所以当 $\varepsilon=\frac{\alpha}{2}$时,得到近似空间分数阶导数的二阶算子如[9]下:

$$\frac{\partial^\alpha u(x_i,t_{n+1})}{\partial_+x^\alpha}=\frac{1}{h^\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{i}g_kv_{i-k}^{n+1}+\frac{\alpha}{2}\sum_{k=0}^{i+1}g_kv_{i-k+1}^{n+1}\right]+O(h^2),$$
$$\frac{\partial^\alpha u(x_i,t_{n+1})}{\partial_-x^\alpha}=\frac{1}{h^\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{K-i}g_kv_{i+k}^{n+1}+\frac{\alpha}{2}\sum_{k=0}^{K-i+1}g_kv_{i+k-1}^{n+1}\right]+O(h^2)。\tag{19}$$

对于对流项的离散,由于 $p(x)\geqslant0$,采用经典的向后迎风格式并且保留其截断误差项中的二阶导数项,并采用第 n 时刻的中心差分做为修正[10],即

$$\frac{\partial u(x_i,t_{n+1})}{\partial x}=\frac{u_i^{n+1}-u_{i-1}^{n+1}}{h}+\frac{1}{2h}(u_{i+1}^n-2u_i^n+u_{i-1}^n)+O(\tau+h^2)。\tag{20}$$

对于方程(1)～(5),时间方向采用向前 Euler 方法离散,并结合公式(19),(20)式即有:

$$\frac{v_i^{n+1}-v_i^n}{\Delta t}=-p_i\left[\frac{v_i^{n+1}-v_{i-1}^{n+1}}{h}+\frac{1}{2h}(v_{i+1}^n-2v_i^n+v_{i-1}^n)\right]+$$
$$\frac{c_{+,i}}{h^\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{i}g_kv_{i-k}^{n+1}+\frac{\alpha}{2}\sum_{k=0}^{i+1}g_kv_{i-k+1}^{n+1}\right]+$$
$$\frac{c_{-,i}}{h^\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{K-i}g_kv_{i+k}^{n+1}+\frac{\alpha}{2}\sum_{k=0}^{K-i+1}g_kv_{i+k-1}^{n+1}\right]+s_i^{n+1}。\tag{21}$$

令 $\zeta_i=\frac{p_i\Delta t}{h}$,$\xi_i=\frac{c_{+,i}\Delta t}{h^\alpha}$,$\eta_i=\frac{c_{-,i}\Delta t}{h^\alpha}$,将上式整理得:

$$\left[1+\zeta_i-\xi_i\left(1-\frac{\alpha}{2}\right)g_0-\xi_i\frac{\alpha}{2}g_1-\eta_i\left(1-\frac{\alpha}{2}\right)g_0-\eta_i\frac{\alpha}{2}g_1\right]v_i^{n+1}-$$
$$\left[\xi_i\frac{\alpha}{2}g_0+\eta_i\left(1-\frac{\alpha}{2}\right)g_1+\eta_i\frac{\alpha}{2}g_2\right]v_{i+1}^{n+1}-\left[\zeta_i+\xi_i\left(1-\frac{\alpha}{2}\right)g_1+\xi_i\frac{\alpha}{2}g_2+\eta_i\frac{\alpha}{2}g_0\right]v_{i-1}^{n+1}-$$
$$\xi_i\sum_{k=3}^{i+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]v_{i-k+1}^{n+1}-\eta_i\sum_{k=3}^{K-i+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]v_{i+k-1}^{n+1}=$$
$$(1+\zeta_i)v_i^n-\frac{\zeta_i}{2}v_{i+1}^n-\frac{\zeta_i}{2}v_{i-1}^n+\Delta t s_i^{n+1}。\tag{22}$$

考虑了方程(1)的初边值条件,可以把(21)式写成如下的矩阵形式:

$$\boldsymbol{A}\boldsymbol{U}^{n+1}=\boldsymbol{B}\boldsymbol{U}^n+\Delta t\boldsymbol{S}^{n+1},\tag{23}$$

其中:

$$\boldsymbol{U}^{n+1}=[v_1^{n+1},v_2^{n+1},\cdots,v_{K-1}^{n+1}]^{\mathrm{T}},$$
$$\boldsymbol{S}^{n+1}=[S_1^{n+1},S_2^{n+1},\cdots,S_{K-1}^{n+1}]^{\mathrm{T}},$$
$$\boldsymbol{A}=(A_{ij})_{(K-1)\times(K=1)},\boldsymbol{B}=(B_{ij})_{(K-1)\times(K=1)},\tag{24}$$

$$A_{ij}=\begin{cases}1+\zeta_i-\xi_i\left(1-\frac{\alpha}{2}\right)g_0-\xi_i\frac{\alpha}{2}g_1-\eta_i\left(1-\frac{\alpha}{2}\right)g_0-\eta_i\frac{\alpha}{2}g_1,j=i,\\-\xi_i\frac{\alpha}{2}g_0-\eta_i\left(1-\frac{\alpha}{2}\right)g_1-\eta_i\frac{\alpha}{2}g_2,j=i+1,\\-\zeta_i-\xi_i\left(1-\frac{\alpha}{2}\right)g_1-\xi_i\frac{\alpha}{2}g_2-\eta_i\frac{\alpha}{2}g_0,j=i-1,\\-\xi_i\left[\left(1-\frac{\alpha}{2}\right)g_{i-j}+\frac{\alpha}{2}g_{i-j+1}\right],j\leqslant i-2,\\-\eta_i\left[\left(1-\frac{\alpha}{2}\right)g_{j-i}+\frac{\alpha}{2}g_{j-i+1}\right],j\geqslant i+2。\end{cases}\tag{25}$$

$$B_{ij}=\begin{cases}1+\zeta_i,j=i,\\-\frac{\zeta_i}{2},j=i+1,\\-\frac{\zeta_i}{2},j=i-1,\\0,\text{其他}。\end{cases}\tag{26}$$

3 稳定性分析

定理3 若$\frac{\sqrt{17}-1}{2}\leqslant\alpha\leqslant 2$,方程组(23)的系数矩阵$\boldsymbol{A}$是对角占优的。

证明 当$1<\alpha<2$,且$\varepsilon=\frac{\alpha}{2}$时,$\alpha\varepsilon-(1-\varepsilon)>0$自然成立,所以

$$Aii=1+\zeta_i-\xi_i(1-\varepsilon)g_0-\xi_i\varepsilon g_1-\eta_i(1-\varepsilon)g_0-\eta_i\varepsilon g_1>0。\tag{27}$$

若$g_2\varepsilon+g_1(1-\varepsilon)\geqslant 0$,由(9)式可得:

$$\frac{\alpha^2}{2}\varepsilon+\frac{\alpha}{2}\varepsilon-\alpha\geqslant 0\Rightarrow\alpha\geqslant\frac{2-\varepsilon}{\varepsilon}。\tag{28}$$

当$\varepsilon=\frac{\alpha}{2}$时,(28)式等价于$\frac{\alpha^2}{2}+\frac{\alpha}{2}-2\geqslant 0$且$1<\alpha<2$,可以得到此不等式的解为:

$$\frac{\sqrt{17}-1}{2}\leqslant\alpha\leqslant 2。\tag{29}$$

综合上面分析,如果$\frac{\sqrt{17}-1}{2}\leqslant\alpha\leqslant 2$,可以得到如下结论:

$$A_{i,i+1}=-\xi_i\varepsilon g_0-\eta_i(1-\varepsilon)g_1-\eta_i\varepsilon g_2=-\xi_i\varepsilon g_0-\eta_i[(1-\varepsilon)g_1+\varepsilon g_2]\leqslant 0,$$
$$A_{i,i-1}=-\xi_i(1-\varepsilon)g_1-\xi_i\varepsilon g_2-\eta_i\varepsilon g_0=-\xi_i[(1-\varepsilon)g_1+\varepsilon g_2]-\eta_i\varepsilon g_0\leqslant 0,$$

$$A_{i,j}=-\xi_i[(1-\varepsilon)g_{i-j}+\varepsilon g_{i-j+1}]\leqslant 0,j\leqslant i-2,$$
$$A_{i,j}=\eta_i[(1-\varepsilon)g_{j-i}+\varepsilon g_{j-i+1}]\leqslant 0,j\geqslant i+2。\tag{30}$$

再利用(9)式,经过简单的推导可得:$A_{i,i}\geqslant\sum\limits_{K-1}|A_{i,j}|$,所以矩阵$\boldsymbol{A}$是对角占优的。

定理4　若$\frac{\sqrt{17}-1}{2}\leqslant\alpha\leqslant 2$,则对于解带有初边值问题的方程(1)～(5)定义的二阶隐式有限差分格式(20)是无条件稳定的。

证明　设$v_i^n,\tilde{v}_i^n(i=1,2,\cdots,K-1;n=0,1,2,\cdots,N-1)$是有限差分格式(21)的数值解,令:$\varepsilon_i^n=\tilde{v}_i^n-v_i^n,E^n=(\varepsilon_i^1,\varepsilon_i^2,\cdots,\varepsilon_{K-1}^n)$,显然,$AE^1=BE^0,AE^{n+1}=BE^n$。现在用数学归纳法证明:$\|E^n\|_\infty\leqslant(1+K_1\tau)\|E^0\|_\infty,n=0,1,\cdots,N-1$。

当$n=1$时,设$\varepsilon_l^1=\max\limits_{1\leqslant i\leqslant K-1}|\varepsilon_i^1|$。当$\frac{\sqrt{17}-1}{2}\leqslant\alpha\leqslant 2$时,(27),(30)式成立且$\zeta_i\geqslant 0,\eta_i\geqslant 0,\xi_i\geqslant 0$,再由$g_k$的性质(9)可得:

$$\begin{aligned}\|E^1\|_\infty=|\varepsilon_l^1|\leqslant&|\varepsilon_l^1|-\eta_l\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{K-l}g_k|\varepsilon_l^1|-\eta_l\frac{\alpha}{2}\sum_{k=0}^{K-l+1}g_k|\varepsilon_l^1|-\xi_l\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{i}g_k|\varepsilon_l^1|-\xi_l\frac{\alpha}{2}\sum_{k=0}^{l+1}g_k|\varepsilon_l^1|\leqslant\\&\left(1+\zeta_l-\xi_l\left(1-\frac{\alpha}{2}\right)g_0-\xi_l\frac{\alpha}{2}g_1-\eta_l\left(1-\frac{\alpha}{2}\right)g_0-\eta_l\frac{\alpha}{2}g_1\right)|\varepsilon_l^1|-\\&\left(\xi_l\frac{\alpha}{2}g_0+\eta_l\left(1-\frac{\alpha}{2}\right)g_1+\eta_l\frac{\alpha}{2}g_2\right)|\varepsilon_{l+1}^1|-\left(\xi_l\left(1-\frac{\alpha}{2}\right)g_1+\xi_l\frac{\alpha}{2}g_2+\eta_l\frac{\alpha}{2}g_0\right)|\varepsilon_{l-1}^1|-\\&\xi_l\sum_{k=3}^{l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]|\varepsilon_{l-k+1}^1|-\eta_l\sum_{k=3}^{K-l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]|\varepsilon_{l+k-11}^1|=\\&(1+\zeta_l)|\varepsilon_l^0|-\frac{\zeta_l}{2}|\varepsilon_{l+1}^0|-\frac{\zeta_l}{2}|\varepsilon_{l-1}^0|\leqslant(1+\zeta_l)|\varepsilon_l^0|+\frac{\zeta_l}{2}|\varepsilon_{l+1}^0|+\frac{\zeta_l}{2}|\varepsilon_{l-1}^0|\leqslant\\&(1+2\zeta_l)|\varepsilon_l^0|=(1+K_1\tau)\|E^0\|_\infty\end{aligned}$$

其中K_1为正常数。

假设$\|E^j\|_\infty\leqslant(1+K_1\tau)\|E^0\|_\infty,j=1,2,\cdots,n-1$。则当$j=n$时,令$|\varepsilon_l^n|=\max\limits_{1\leqslant i\leqslant K-1}|\varepsilon_i^n|,1\leqslant l\leqslant K-1$。同理可得:

$$\begin{aligned}\|E^n\|_\infty=|\varepsilon_l^n|\leqslant&|\varepsilon_l^n|-\eta_l\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{K-l}g_k|\varepsilon_l^n|-\eta_l\frac{\alpha}{2}\sum_{k=0}^{K-l+1}g_k|\varepsilon_l^n|-\xi_l\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{l}g_k|\varepsilon_l^n|-\xi_l\frac{\alpha}{2}\sum_{k=0}^{l+1}g_k|\varepsilon_l^n|\leqslant\\&\left(1+\zeta_l-\xi_l\left(1-\frac{\alpha}{2}\right)g_0-\xi_l\frac{\alpha}{2}g_1-\eta_l\left(1-\frac{\alpha}{2}\right)g_0-\eta_l\frac{\alpha}{2}g_1\right)|\varepsilon_l^n|-\\&\left(\xi_l\left(1-\frac{\alpha}{2}\right)g_1+\xi_l\frac{\alpha}{2}g_2+\eta_l\frac{\alpha}{2}g_0+\zeta_l\right)|\varepsilon_{l-1}^n|-\left(\xi_l\frac{\alpha}{2}g_0-\eta_l\left(1-\frac{\alpha}{2}\right)g_1-\eta_l\frac{\alpha}{2}g_2\right)|\varepsilon_{l+1}^n|-\\&\xi_l\sum_{k=3}^{l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]|\varepsilon_{l-k+1}^n|-\eta_l\sum_{k=3}^{K-l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]|\varepsilon_{l-k+1}^n|=\\&(1+\zeta_l)|\varepsilon_l^{n-1}|-\frac{\zeta_l}{2}|\varepsilon_{l+1}^{n-1}|-\frac{\zeta_l}{2}|\varepsilon_{l-1}^{n-1}|\leqslant(1+2\zeta_l)|\varepsilon_l^{n-1}|=(1+K_1\tau)\|E^{n-1}\|_\infty\leqslant\\&(1+K_1\tau)(1+K_1\tau)^{n-1}\|E^0\|_\infty\leqslant e^{K_1n\tau}\|E^0\|_\infty。\end{aligned}$$

其中,$e^{K_1n\tau}\leqslant e^{K_1T}(n\tau\leqslant T)$。所以,存在正常数$M$,使得$\|E^n\|_\infty\leqslant M\|E^0\|_\infty$,定理得证。　证毕

4 收敛性分析

定理5　设$u(x_i,t_n)$是方程(1)～(4)的准确解,v_i^n是二阶加权有限差分格式(21)的计算值,则当$\varepsilon=\frac{\alpha}{2}$且$\frac{\sqrt{17}-1}{2}\leqslant\alpha\leqslant 2$时,使得$\max\limits_{1\leqslant i\leqslant K-1}|v_i^n-u(x_i,t_n)|\leqslant C(\tau+h^2)$,$n=1,2,\cdots,\frac{T}{\tau}$,其中$C$是正常数。

证明　令$e_i^n=v_i^n-u(x_i,t_n)$,且R_i^n表示点(x_i,t^n)的截断误差,则

$$R_i^n=\left(\frac{\partial u}{\partial t}+p(x)\frac{\partial u}{\partial x}-c_+\frac{\partial^\alpha u}{\partial_+x^\alpha}-c_-\frac{\partial^\alpha u}{\partial_-x^\alpha}\right)(x_i,t_{n+1})-\frac{u(x_i,t_{n+1})-u(x_i,t_n)}{\Delta t}-$$

$$p(x_i)\left\{\frac{u(x_i,t_{n+1})-u(x_{i-1},t_{n+1})}{h}+\frac{1}{2h}\left[u(x_{i+1},t_n)-2u(x_i,t_n)+u(x_{i-1},t_n)\right]\right\}+$$
$$\frac{c_+(x_i,t_{n+1})}{h^\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{l}g_k u(x_{i-k},t_{n+1})+\frac{\alpha}{2}\sum_{k=0}^{l+1}g_k u(x_{i-k+1},t_{n+1})\right]+$$
$$\frac{c_-(x_i,t_{n+1})}{h^\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{K-l}g_k u(x_{i+k},t_{n+1})+\frac{\alpha}{2}\sum_{k=0}^{K-l+1}g_k u(x_{i+k-1},t_{n+1})\right]=O(\tau+h^2)。$$

设 $E^n=[e_1^n,e_2^n,\cdots,e_{K-1}^n]^{\mathrm{T}}$, $R^n=[R_1^n,R_2^n,\cdots,R_{K-1}^n]^{\mathrm{T}}$,直接计算就可以得到

$$\left(1+\zeta_i-\xi_i\left(1-\frac{\alpha}{2}\right)g_0-\xi_i\frac{\alpha}{2}g_1-\eta_i\left(1-\frac{\alpha}{2}\right)g_0-\eta_i\frac{\alpha}{2}g\right)e_i^{n+1}-$$
$$\left(\xi_i\frac{\alpha}{2}g_0+\eta_i\left(1-\frac{\alpha}{2}\right)g_1+\eta_i\frac{\alpha}{2}g_2\right)e_{i+1}^{n+1}-\left(\xi_i\left(1-\frac{\alpha}{2}\right)g_1+\xi_i\frac{\alpha}{2}g_2+\eta_i\frac{\alpha}{2}g_0\right)e_{i-1}^{n+1}-$$
$$\xi_i\sum_{k=3}^{l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]e_{i-k+1}^{n+1}-\eta_i\sum_{k=3}^{K-l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]e_{i+k-1}^{n+1}=$$
$$(1+\zeta_i)e_i^n-\frac{\zeta_i}{2}e_{i+1}^n-\frac{\zeta_i}{2}e_{i-1}^n+\tau R_i^n$$

接下来,用数学归纳法证明下列不等式:

$\|E^n\|_\infty\leqslant C(1+L\tau)^{n-1}n\tau(\tau+h^2)$, $n=0,1,\cdots,N-1$,其中 L 为正常数。

设 $n=1$ 时,令

$$\|E^1\|_\infty=|e_l^1|\leqslant|e_l^1|-\eta_l\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{K-l}g_k|e_l^1|-\eta_l\frac{\alpha}{2}\sum_{k=0}^{k-l+1}g_k|e_l^1|-$$
$$\xi_l\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{l}g_k|e_l^1|-\xi_l\frac{\alpha}{2}\sum_{k=0}^{k-l+1}g_k|e_l^1|\leqslant$$
$$\left(1+\zeta_l-\xi_l\left(1-\frac{\alpha}{2}\right)g_0-\xi_l\frac{\alpha}{2}g_1-\eta_l\left(1-\frac{\alpha}{2}\right)g_0-\eta_l\frac{\alpha}{2}g\right)|e_l^1|-$$
$$\left(\xi_l\frac{\alpha}{2}g_0+\eta_l\left(1-\frac{\alpha}{2}\right)g_1+\eta_l\frac{\alpha}{2}g_2\right)e_{l+1}^{n+1}-\left(\xi_l\left(1-\frac{\alpha}{2}\right)g_1+\xi_l\frac{\alpha}{2}g_2+\eta_l\frac{\alpha}{2}g_0\right)|e_{l-1}^1|-$$
$$\xi_l\sum_{k=3}^{l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]|e_{l-k+1}^1|-\eta_l\sum_{k=3}^{K-l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]|e_{l+k-1}^1|\leqslant$$
$$\left|\left(1+\zeta_l-\xi_l\left(1-\frac{\alpha}{2}\right)g_0-\xi_l\frac{\alpha}{2}g_1-\eta_l\left(1-\frac{\alpha}{2}\right)g_0-\eta_l\frac{\alpha}{2}g\right)e_l^1-\right.$$
$$\left(\xi_l\frac{\alpha}{2}g_0+\eta_l\left(1-\frac{\alpha}{2}\right)g_1+\eta_l\frac{\alpha}{2}g_2\right)e_{l+1}^1-\left(\xi_l\left(1-\frac{\alpha}{2}\right)g_1+\xi_l\frac{\alpha}{2}g_2+\eta_l\frac{\alpha}{2}g_0\right)e_{l-1}^1-$$
$$\left.\xi_i\sum_{k=3}^{l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]e_{l-k+1}^1-\eta_l\sum_{k=3}^{K-l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]e_{l+k-1}^1\right|=$$
$$\left|(1+\zeta_l)e_l^0-\frac{\zeta_l}{2}e_{l+1}^0-\frac{\zeta_l}{2}e_{l-1}^0+C\tau(\tau+h^2)\right|=C\tau(\tau+h^2)=C(1+L\tau)^0\tau(\tau+h^2)。$$

设 $\|E^{n-1}\|_\infty\leqslant C(1+L\tau)^{n-2}(n-1)\tau(\tau+h^2)$, $n=0,1,\cdots,N-1$,且 $\|E^n\|_\infty=|e_l^n|=\max\limits_{1\leqslant i\leqslant K-1}|e_i^n|$, $(1\leqslant l\leqslant K-1)$,则得到:

$$\|E^n\|_\infty=|e_l^n|\leqslant|e_l^n|-\eta_l\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{K-l}g_k|e_l^n|-\eta_l\frac{\alpha}{2}\sum_{k=0}^{k-l+1}g_k|e_l^n|-\xi_l\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{l}g_k|e_l^n|-$$
$$\xi_l\frac{\alpha}{2}\sum_{k=0}^{k-l+1}g_k|e_l^n|\leqslant\left(1+\zeta_l-\xi_l\left(1-\frac{\alpha}{2}\right)g_0-\xi_l\frac{\alpha}{2}g_1-\eta_l\left(1-\frac{\alpha}{2}\right)g_0-\eta_l\frac{\alpha}{2}g\right)|e_l^n|-$$
$$\left(\xi_l\frac{\alpha}{2}g_0+\eta_l\left(1-\frac{\alpha}{2}\right)g_1+\eta_l\frac{\alpha}{2}g_2\right)|e_{l+1}^n|-\left(\xi_l\left(1-\frac{\alpha}{2}\right)g_1+\xi_l\frac{\alpha}{2}g_2+\eta_l\frac{\alpha}{2}g_0\right)|e_{l-1}^n|-$$
$$\xi_l\sum_{k=3}^{l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]|e_{l-k+1}^n|-\eta_l\sum_{k=3}^{K-l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]|e_{l+k-1}^n|\leqslant$$
$$\left|\left(1+\zeta_l-\xi_l\left(1-\frac{\alpha}{2}\right)g_0-\xi_l\frac{\alpha}{2}g_1-\eta_l\left(1-\frac{\alpha}{2}\right)g_0-\eta_l\frac{\alpha}{2}g\right)e_l^n-\right.$$

$$\left(\xi_l\,\frac{\alpha}{2}g_0+\eta_l\left(1-\frac{\alpha}{2}\right)g_1+\eta_l\,\frac{\alpha}{2}g_2\right)e_{l+1}^n-\left(\xi_l\left(1-\frac{\alpha}{2}\right)g_1+\xi_l\,\frac{\alpha}{2}g_2+\eta_l\,\frac{\alpha}{2}g_0\right)e_{l-1}^n-$$

$$\xi_i\sum_{k=3}^{l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]e_{i-k+1}^n-\eta_i\sum_{k=3}^{K-l+1}\left[\left(1-\frac{\alpha}{2}\right)g_{k-1}+\frac{\alpha}{2}g_k\right]e_{i+k-1}^n\Bigg|=$$

$$\left|(1+\zeta_l)e_l^{n-1}-\frac{\zeta_l}{2}e_{l+1}^{n-1}-\frac{\zeta_l}{2}e_{l-1}^{n-1}+C\tau(\tau+h^2)\right|\leqslant(1+\zeta_l)\left|e_l^{n-1}\right|+\frac{\zeta_l}{2}\left|e_{l+1}^{n-1}\right|+$$

$$\frac{\zeta_l}{2}\left|e_{l-1}^{n-1}\right|+C\tau(\tau+h^2)\leqslant(1+2\zeta_l)\ \|E^{n-1}\|_\infty+C\tau(\tau+h^2)\leqslant$$

$$(1+L\tau)C(1+L\tau)^{n-2}(n-1)\tau(\tau+h^2)+C\tau(\tau+h^2)(1+L\tau)^{n-1}=C(1+L\tau)^{n-1}n\tau(\tau+h^2)。$$

因为$(1+L\tau)^{n-1}\leqslant e^{L(n-1)\tau}\leqslant e^{LT}\,(n\tau\leqslant T)$,所以存在正常数 M,使得

$$\max_{1\leqslant i\leqslant K-1}\left|v_i^n-u(x_i,t_n)\right|\leqslant M(\tau+h^2),n=1,2,\cdots,N-1;i=1,2,\cdots,K-1。$$ 证毕

5 数值算例

考虑带有如下初边值问题的变系数双边空间分数阶偏微分方程:

$$\frac{\partial u(x,t)}{\partial t}=-p(x)\frac{\partial u(x,t)}{\partial x}+c_+(x,t)\frac{\partial^{1.8}u(x,t)}{\partial_+x^{1.8}}+c_-(x,t)\frac{\partial^{1.8}u(x,t)}{\partial_-x^{1.8}}+s(x,t),$$

$$u(x,0)=x^2(x-2)^2,0<x<2,$$

$$u(0,t)=u(2,t)=0,t>0。\tag{31}$$

其中:

$$p(x)=x,c_+(x,t)=\Gamma(3-\alpha)x^{1.8},c_-(x,t)=\Gamma(3-\alpha)(2-x)^{1.8},$$

$$s(x,t)=-e^{-t}\left[8(2-x)^2-24\,\frac{\Gamma(3-\alpha)}{\Gamma(4-\alpha)}(x^3+(2-x)^3)+24\,\frac{\Gamma(3-\alpha)}{\Gamma(5-\alpha)}\times\right.$$

$$\left.(x^4+(2-x)^4)+12x^3-4x^4\right]+e^{-t}x^2(2-x)^2,\alpha=1.8。\tag{32}$$

此方程的精确解为 $u(x,t)=e^{-t}x^2(2-x)^2$。

表 1 给出了此方程当 $t=1.0$ 时刻由本文提出的隐式有限差分方法得到最大误差 $E_1=\|e\|_{l^\infty}=\max\limits_{1\leqslant i\leqslant K-1}\left|u(x_i,t^n)-v_i^n\right|$,$E_2=\|e\|_{l^2}=\left(\sum\limits_{i=1}^{K-1}\left|u(x_i,t_n)-v_i^n\right|^2h\right)^{\frac{1}{2}}$ 和收敛阶 $Rate=\log_2(E^n/E^{n+1})/\log_2(h^n/h^{n+1})$,$E^n$,$E^{n+1}$代表相邻两次的最大误差,$h^n$,$h^{n+1}$代表相邻两次的步长。从表中可以看到,本文所构造的加权隐式有限差分格式具有二阶精度,需要说明的是时间步长都取空间步长的平方,这样时间方向不会影响到空间的二阶收敛性。

表 1　用二阶隐式有限差分求解在 $t=1.0$ 时的最大误差和收敛阶

$(\Delta t,h)$	$\|e\|_{l^\infty}$	$Rate$	$\|e\|_{l^2}$	$Rate$
$\left(\frac{1}{10^2},\frac{1}{10}\right)$	2.486 8E−02	—	2.520 6E−02	—
$\left(\frac{1}{20^2},\frac{1}{20}\right)$	6.272 8E−03	1.98	6.323 3E−03	1.995
$\left(\frac{1}{30^2},\frac{1}{30}\right)$	2.793 0E−03	1.996	2.835 6E−03	1.98
$\left(\frac{1}{40^2},\frac{1}{40}\right)$	1.577 9E−03	1.98	1.601 6E−03	1.99
$\left(\frac{1}{50^2},\frac{1}{50}\right)$	9.723 3E−04	2.17	9.896 0E−04	2.16
$\left(\frac{1}{60^2},\frac{1}{60}\right)$	6.631 3E−04	2.099	6.735 6E−04	2.11
$\left(\frac{1}{70^2},\frac{1}{70}\right)$	4.727 2E−04	2.19	4.778 1E−04	2.227
$\left(\frac{1}{80^2},\frac{1}{80}\right)$	2.834 2E−04	3.83	2.907 9E−04	3.72

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A Kind of Second-order Implicit Finite Difference Methods for Two-sided Space Fractional Advection-dispersion Partial Differential Equations with Variable Coefficient

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Abstract: A kind of implicit finite difference schemes for two-sided space fractional advection-dispersion partial differential equations with variable coefficient is introduced. This kind of schemes' unconditionally stable and convergence rate $O(\Delta t+h^2)$ with fractional derivative α belonging to $\frac{\sqrt{17}-1}{2}\leqslant\alpha\leqslant 2$ are proved. Numerical examples are given to show the efficiency and the convergence rate of presented schemes.

Key words: two-sided space fractional partial differential equations with variable coefficient; finite difference scheme; unconditionally stable; convergence rate

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