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一个新孤子方程族的 Darboux 变换及其精确解*

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摘要：从一个新的 2×2 矩阵谱问题出发导出了一族非线性孤子方程，针对前两个非平凡的孤子方程，通过谱问题的基解矩阵，利用其 Lax 对的规范变换，得到了孤子方程的 Darboux 变换。作为应用，给出了孤子方程的一些精确解，并作出了孤子图，有助于对方程所描述的自然现象进行分析和研究。

关键词：零曲率方程；Darboux 变换；规范变换；精确解

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孤子理论的发现是近年来非线性理论中的重大进展之一，它既能反映一类非常稳定的自然现象，体现一大类非线性物质相互作用的若干本质特征，并为许多应用问题(如光纤通讯)提供了启示。另一方面又为非线性偏微分方程提供了求显式解的方法，因而受到物理学家和数学家的重视。孤子理论已经成为研究非线性方程的主要理论之一，目前在孤子理论中蕴藏着一系列构造精确解的方法，如：反散射方法[1-3]，非线性化方法[4-5]，Hirota 双线性方法[6-7]，齐次平衡法[8-9]，Backlund 变换方法[10-11]，Darboux 变换方法[12-13]等。这些方法的发现及其应用，大量的非线性偏微分方程得以成功求解，从而对非线性科学的发展和应用具有十分重要意义。其中 Darboux 变换法在 19 世纪研究 Sine-Gordon 方程与伪球面的有关问题时就已经出现[14-16]，它从平凡解出发，可以得到孤子方程的精确解，是相当有效而简明的方法。

本文研究孤子方程的 Darboux 变换方法，通过研究一个新的含三个位势的 2×2 矩阵谱问题，导出一族非线性孤子方程。针对前两个非平凡的孤子方程，通过矩阵谱问题的基解矩阵，利用其 Lax 对的规范变换，得到了孤子方程的 Darboux 变换。作为应用，求出了孤子方程的一些精确解，并作出了孤子图。

1 Lenard 序列与非线性孤子族

考虑 2×2 特征值谱问题

$$\phi_x=\boldsymbol{U}\phi,\ \phi=\begin{pmatrix}p\\q\end{pmatrix},\tag{1}$$

其中，$\boldsymbol{U}=\begin{pmatrix}u & -\alpha\varepsilon\lambda^2+\alpha v\lambda+w\\ \varepsilon\lambda+v & -u\end{pmatrix}$，这里 u,v,w 是位势函数，λ 是谱参数，α,ε 是常数。

令

$$\phi_t=\boldsymbol{V}\phi,\tag{2}$$

其中，$\boldsymbol{V}=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}=\sum_{j=0}^{+\infty}\begin{pmatrix}a_j & b_j\\ c_j & -a_j\end{pmatrix}\lambda^{-j}$。

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由驻定零曲率方程

$$\boldsymbol{V}_x=[\boldsymbol{U},\boldsymbol{V}], \tag{3}$$

比较方程两端 λ 同次幂系数可得

$$\begin{cases}a_0=0,b_0=-\alpha\varepsilon,c_0=0,a_1=0,b_1=\alpha v,\\ c_1=\varepsilon,a_2=u,b_2=\dfrac{\varepsilon w-\alpha v^2}{2\varepsilon},c_2=v,\\ a_3=\dfrac{v_x}{2\varepsilon},b_3=\dfrac{-\varepsilon u_x+\alpha v^3-\varepsilon u^2}{2\varepsilon^2},c_3=\dfrac{\alpha v^2+\varepsilon w}{2\alpha\varepsilon},\cdots,\end{cases} \tag{4}$$

并得到关系式

$$\begin{cases}a_{jx}=wc_j-vb_j+\alpha vc_{j+1}-\varepsilon b_{j+1}-\alpha\varepsilon c_{j+2},\\ b_{jx}=2ub_j-2wa_j-2\alpha va_{j+1}+2\alpha\varepsilon a_{j+2},\\ c_{jx}=2va_j-2uc_j+2\varepsilon a_{j+1}。\end{cases} \tag{5}$$

定义 $g_j=(2a_{j+1},\alpha c_{j+2}+b_{j+1},c_{j+1})^{\mathrm{T}}$,得到 Lenard 递推方程

$$\boldsymbol{K}g_{j-1}=\boldsymbol{J}g_j,j\geqslant 0, \tag{6}$$

其中

$$\boldsymbol{K}=\begin{pmatrix}\frac{1}{2}\partial & v & -w\\ -v & 0 & \partial+2u\\ w & \partial-2u & 0\end{pmatrix},\quad \boldsymbol{J}=\begin{pmatrix}0 & -\varepsilon & 2\alpha v\\ \varepsilon & 0 & 0\\ -2\alpha v & 0 & 2\alpha\partial\end{pmatrix}, \tag{7}$$

$\boldsymbol{K},\boldsymbol{J}$ 是两个斜对称算子。定义 $g_{-1}=(0,2\alpha v,\varepsilon)^{\mathrm{T}}$,则 $\boldsymbol{J}g_{-1}=0$,利用递推关系(5),则所有的 $g_j,j\geqslant 0$ 均可唯一确定。

下面引入(1)式的辅助谱问题

$$\phi_{t_n}=\boldsymbol{V}^{(n)}\phi,\boldsymbol{V}^{(n)}=\begin{pmatrix}V_{11}^{(n)} & V_{12}^{(n)}\\ V_{21}^{(n)} & -V_{11}^{(n)}\end{pmatrix},n\geqslant 1, \tag{8}$$

其中

$$V_{11}^{(n)}=\sum_{j=2}^{n}a_j\lambda^{n-j},$$

$$V_{12}^{(n)}=\sum_{j=1}^{n-1}b_j\lambda^{n-j}-\varepsilon\alpha\lambda^n+b_n+\alpha c_{n+1},$$

$$V_{21}^{(n)}=\sum_{j=1}^{n}c_j\lambda^{n-j}。$$

由零曲率方程 $\boldsymbol{U}_{t_n}-\boldsymbol{V}_x^{(n)}+[\boldsymbol{U},\boldsymbol{V}^{(n)}]=0$,即得孤子族 $(u,v,w)_{t_n}^{\mathrm{T}}=\boldsymbol{J}g_{n-1}=X_{n-2},n\geqslant 2$。令 $t_2=x,t_3=y,t_4=t$,则有

$$\begin{pmatrix}u\\ v\\ w\end{pmatrix}_x=\boldsymbol{X}_0=\begin{pmatrix}u_x\\ v_x\\ w_x\end{pmatrix},\quad \begin{pmatrix}u\\ v\\ w\end{pmatrix}_y=\boldsymbol{X}_1,\quad \begin{pmatrix}u\\ v\\ w\end{pmatrix}_t=\boldsymbol{X}_2, \tag{9}$$

于是得前两个非平凡的(1+1)-维孤子方程

$$\begin{pmatrix}u\\ v\\ w\end{pmatrix}_y=\frac{1}{2\alpha\varepsilon^2}\begin{pmatrix}\alpha\varepsilon v_{xx}+\alpha\varepsilon v^2w+2\alpha^2v^4-\varepsilon^2w^2-2\alpha\varepsilon u_xv\\ 2\varepsilon^2uw+\varepsilon^2w_x+2\alpha\varepsilon uv^2\\ -4\alpha\varepsilon uvw-4\alpha^2uv^3-2\alpha\varepsilon vw_x+\partial(2\alpha\varepsilon u^2+2\alpha^2v^3+4\alpha\varepsilon vw-2\alpha\varepsilon u_x)\end{pmatrix}, \tag{10}$$

$$\begin{pmatrix}u\\ v\\ w\end{pmatrix}_t=\frac{1}{2\alpha\varepsilon^2}\begin{pmatrix}\partial(\varepsilon uw+\frac{1}{2}\varepsilon w_x+\alpha uv^2)-\alpha vv_{xx}-\varepsilon vw^2+2\alpha u^2v^2+2\alpha v^3w-\varepsilon u^2w+\varepsilon u_xw\\ \partial(2\varepsilon vw+\alpha v^3-\varepsilon u_x)+2\varepsilon uvw-\varepsilon vw_x+2\varepsilon u^3\\ \partial(-\alpha v_{xx}+2\alpha u^2v+3\alpha v^2w+\varepsilon w^2)+\varepsilon ww_x+2\alpha uv_{xx}-4\alpha u^3v-4\alpha uv^2w\end{pmatrix}。 \tag{11}$$

2 Darboux 变换

方程(10)和(11)谱问题是(1)式，相应的辅助谱问题为

$$\phi_y = \boldsymbol{V}_1 \phi, \tag{12}$$

其中

$$\boldsymbol{V}_1 = \begin{pmatrix} V_1^{(11)} & V_1^{(12)} \\ V_1^{(21)} & V_1^{(22)} \end{pmatrix},$$

$$V_1^{(11)} = u\lambda + \frac{1}{2\varepsilon} v_x, V_1^{(12)} = -\alpha\varepsilon\lambda^3 + \alpha v \lambda^2 + \left(\frac{w}{2} - \frac{\alpha v^2}{2\varepsilon}\right)\lambda + \frac{vw}{\varepsilon} + \frac{\alpha v^3}{\varepsilon^2} - \frac{u_x}{\varepsilon},$$

$$V_1^{(21)} = \varepsilon\lambda^2 + v\lambda + \frac{w}{2\alpha} + \frac{v^2}{2\varepsilon}, V_1^{(22)} = -u\lambda - \frac{1}{2\varepsilon} v_x 。$$

$$\phi_t = \boldsymbol{V}_2 \phi, \tag{13}$$

其中

$$\boldsymbol{V}_2 = \begin{pmatrix} V_2^{(11)} & V_2^{(12)} \\ V_2^{(21)} & V_2^{(22)} \end{pmatrix},$$

$$V_2^{(11)} = u\lambda^2 + \frac{1}{2\varepsilon} v_x \lambda + \frac{1}{2\varepsilon}\left(\frac{w_x}{2\alpha} + \frac{uv^2}{\varepsilon} + \frac{uw}{\alpha}\right),$$

$$V_2^{(12)} = -\alpha\varepsilon\lambda^4 + \alpha v\lambda^3 + \left(\frac{w}{2} - \frac{\alpha v^2}{2\varepsilon}\right)\lambda^2 + \left(\frac{\alpha v^3}{2\varepsilon^2} - \frac{u_x}{2\varepsilon} - \frac{u^2}{2\varepsilon}\right)\lambda - \frac{1}{2\varepsilon^2} v_{xx} + \frac{w^2}{2\varepsilon\alpha} + \frac{3v^2 w}{2\varepsilon^2} + \frac{u^2 v}{\varepsilon^2},$$

$$V_2^{(21)} = \varepsilon\lambda^3 + v\lambda^2 + \left(\frac{w}{2\alpha} + \frac{v^2}{2\varepsilon}\right)\lambda + \frac{1}{2\alpha\varepsilon}\left(2vw + \frac{\alpha v^3}{\varepsilon} - u_x + u^2\right),$$

$$V_2^{(22)} = -u\lambda^2 - \frac{1}{2\varepsilon} v_x \lambda - \frac{1}{2\varepsilon}\left(\frac{w_x}{2\alpha} + \frac{uv^2}{\varepsilon} + \frac{uw}{\alpha}\right)。$$

考虑谱问题 $\overline{\phi}_x = \overline{\boldsymbol{U}}\,\overline{\phi}, \overline{\phi} = \begin{pmatrix} \overline{\phi_1} \\ \overline{\phi_2} \end{pmatrix}$，其中 $\overline{\boldsymbol{U}}$ 为

$$\overline{\boldsymbol{U}} = \begin{pmatrix} \overline{u} & -\alpha\varepsilon\lambda^2 + \alpha\,\overline{v}\lambda + \overline{w} \\ \varepsilon\lambda + \overline{v} & -\overline{u} \end{pmatrix}。$$

引入谱问题的规范变换

$$\overline{\phi} = \boldsymbol{T}\phi, \tag{14}$$

设

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}, \tag{15}$$

其中 $\boldsymbol{A} = a_1\lambda + a_0, \boldsymbol{B} = b_1\lambda + b_0, \boldsymbol{C} = c_1\lambda + c_0, \boldsymbol{D} = d_1\lambda + d_0, a_j, b_j, c_j, d_j (j = 0, 1)$是关于 x, t 的待定函数。

经过计算可得

$$\overline{v} = v + \frac{d_0 - a_0}{a_1}\varepsilon, \tag{16}$$

$$\begin{cases} \overline{u} = u - \dfrac{\alpha}{a_1} c_0 (v + \overline{v}) + \dfrac{\varepsilon}{a_1} b_0, \\ \overline{u} = \dfrac{a_0}{c_0}\overline{v} - \dfrac{d_0}{c_0} v - u - \dfrac{c_{0,x}}{c_0}, \\ \overline{u} = \dfrac{b_0}{d_0}\overline{v} + u - \dfrac{d_{0,x}}{d_0} - \dfrac{c_0}{d_0}\overline{w}, \end{cases} \tag{17}$$

$$\begin{cases}\overline{w}=\frac{a_{0,x}}{c_0}+\frac{b_0}{c_0}v+\frac{a_0}{c_0}(u-\overline{u}),\\ \overline{w}=\frac{b_{0,x}}{d_0}+\frac{a_0}{d_0}w-\frac{b_0}{d_0}(u+\overline{u}),\\ \overline{w}=-\frac{\alpha}{a_1}c_{0,x}+\frac{\alpha}{a_1}(a_0v-d_0\overline{v})+\frac{\alpha c_0}{a_1}(u+\overline{u}),\end{cases}\tag{18}$$

此时(15)式可化为

$$\boldsymbol{T}=\begin{pmatrix}a_1\lambda+a_0 & -\alpha c_0\lambda+b_0\\ c_0 & a_1\lambda+d_0\end{pmatrix}。\tag{19}$$

设$\varphi(\lambda_j)=(\varphi_1(\lambda_j),\varphi_2(\lambda_j))^{\mathrm{T}}$,$\psi(\lambda_j)=(\psi_1(\lambda_j),\psi_2(\lambda_j))^{\mathrm{T}}$是(1)式的两个基本解,通过(14)式知，存在常数 γ_j 满足

$$\begin{cases}(\boldsymbol{A}\varphi_1(\lambda_j)+\boldsymbol{B}\varphi_2(\lambda_j))-\gamma_j(\boldsymbol{A}\psi_1(\lambda_j)+\boldsymbol{B}\psi_2(\lambda_j))=0,\\ (\boldsymbol{C}\varphi_1(\lambda_j)+\boldsymbol{D}\varphi_2(\lambda_j))-\gamma_j(\boldsymbol{C}\psi_1(\lambda_j)+\boldsymbol{D}\psi_2(\lambda_j))=0,\end{cases}\tag{20}$$

将(20)式可以写成线性系统

$$\begin{cases}a_0+a_1\lambda_j+(b_0-\alpha c_0\lambda_j)\sigma_j=0,\\ c_0+(a_1\lambda_j+d_0)\sigma_j=0,\end{cases}\tag{21}$$

其中

$$\sigma_j=\frac{\varphi_2(\lambda_j)-\gamma_j\psi_2(\lambda_j)}{\varphi_1(\lambda_j)-\gamma_j\psi_1(\lambda_j)},j=1,2。\tag{22}$$

当常数 λ_j,γ_j($\lambda_k\neq\lambda_j$ 若 $k\neq j$)适当选择时，可解得

$$\begin{cases}a_0=\frac{a_1(\lambda_1\sigma_2-\lambda_2\sigma_1)}{\sigma_1-\sigma_2}-\frac{\alpha a_1\sigma_1^2\sigma_2^2(\lambda_1-\lambda_2)^2}{(\sigma_1-\sigma_2)^2},\\ b_0=-\frac{a_1(\lambda_1-\lambda_2)}{\sigma_1-\sigma_2}+\frac{\alpha a_1\sigma_1\sigma_2(\lambda_1-\lambda_2)(\lambda_1\sigma_1-\lambda_2\sigma_2)}{(\sigma_1-\sigma_2)^2},\\ c_0=\frac{a_1\sigma_1\sigma_2(\lambda_1-\lambda_2)}{\sigma_1-\sigma_2},\\ d_0=-\frac{a_1(\lambda_1\sigma_1-\lambda_2\sigma_2)}{\sigma_1-\sigma_2},\end{cases}\tag{23}$$

此时矩阵(19)表明 det T(λ)是 λ 的 2 次多项式且由(21),(23)式知

$$\det T(\lambda)=A(\lambda)D(\lambda)-B(\lambda)C(\lambda)=(a_1\lambda+a_0)(a_1\lambda+d_0)-c_0(-\alpha c_0\lambda+b_0)=a_1^2(\lambda-\lambda_1)(\lambda-\lambda_2),$$

由上式知 λ_j,$j=1,2$ 是 det $T(\lambda)$的根。由(1),(22)式可知,σ_j 满足 Riccati 方程

$$\sigma_{j,x}=\varepsilon\lambda_j+v+(\alpha\varepsilon\lambda_j^2-\alpha v\lambda_j-w)\sigma_j^2-2u\sigma_j,\tag{24}$$

经过复杂的计算,(17)式中 3 个式子可化为

$$\begin{cases}u-\overline{u}=-\frac{\varepsilon}{a_1}b_0+\frac{2\alpha}{a_1}vc_0+\frac{\alpha\varepsilon}{a_1^2}c_0(d_0-a_0),\\ u+\overline{u}=-\frac{a_0-d_0}{c_0}v+\frac{\varepsilon}{a_1c_0}a_0(d_0-a_0)-\frac{c_{0,x}}{c_0},\\ u-\overline{u}=\frac{b_0}{d_0}v-\frac{\varepsilon}{a_1d_0}b_0(d_0-a_0)+\frac{c_0}{d_0}w+\frac{d_{0,x}}{d_0},\end{cases}$$

代入(18)式可得

$$\begin{cases}\overline{w}=\frac{a_{0,x}}{c_0}+\frac{a_0d_{0,x}}{c_0d_0}+\frac{b_0}{c_0}v+\frac{a_0}{d_0}w-\frac{a_0b_0}{c_0d_0}v-\frac{\varepsilon a_0b_0}{a_1c_0d_0}(d_0-a_0),\\ \overline{w}=\frac{b_{0,x}}{d_0}+\frac{b_0c_{0,x}}{c_0d_0}+\frac{b_0}{c_0}v+\frac{a_0}{d_0}w-\frac{a_0b_0}{c_0d_0}v-\frac{\varepsilon a_0b_0}{a_1c_0d_0}(d_0-a_0),\\ \overline{w}=w-\frac{2\alpha}{a_1}c_{0,x}-\frac{2\alpha}{a_1}(d_0-a_0)v-\frac{\alpha\varepsilon}{a_1^2}(d_0-a_0)^2。\end{cases}\tag{25}$$

综上可得

命题 1　在以(19)，(21)式为已知前提下，由 $T_x+TU=\overline{U}T$ 决定的矩阵 $\overline{U}$

$$\overline{U}=\begin{pmatrix}\overline{u} & -\alpha\varepsilon\lambda^2+\alpha\overline{v}\lambda+\overline{w}\\ \varepsilon\lambda+\overline{v} & -\overline{u}\end{pmatrix},$$

与矩阵 U 具有相同的形式，变换(16)，(17)和(18)式将原位势 u,v,w 映射为新位势 $\overline{u},\overline{v},\overline{w}$。

命题 2　在以(19)，(21)式为已知前提下，由 $T_y+TV_1=\overline{V}_1T$ 决定的矩阵 $\overline{V}_1$，

$$\overline{V}_1=\begin{pmatrix}\overline{u}\lambda+\frac{1}{2\varepsilon}\overline{v}_x & -\alpha\varepsilon\lambda^3+\alpha\overline{v}\lambda^2+\left(\frac{\overline{w}}{2}-\frac{\alpha\overline{v}^2}{2\varepsilon}\right)\lambda+\frac{\overline{v}\,\overline{w}}{\varepsilon}+\frac{\alpha\overline{v}^3}{\varepsilon^2}-\frac{\overline{u}_x}{\varepsilon}\varepsilon\\ \lambda^2+\overline{v}\lambda+\frac{\overline{w}}{2\alpha}+\frac{\overline{v}^2}{2\varepsilon} & -\overline{u}\lambda-\frac{1}{2\varepsilon}\overline{v}_x\end{pmatrix},$$

与矩阵 V_1 具有相同形式，变换(16)，(17)和(18)式将原位势 u,v,w 映射为新位势 $\overline{u},\overline{v},\overline{w}$。

命题 3　在以(19)，(21)式为已知前提下，由 $T_t+TV_2=\overline{V}_2T$ 决定的矩阵 $\overline{V}_2$，

$$\overline{V}_2=\begin{pmatrix}\overline{u}\lambda^2+\frac{1}{2\varepsilon}\overline{v}_x\lambda+\frac{1}{2\varepsilon}\left(\frac{\overline{w}_x}{2\alpha}+\frac{\overline{u}^2\overline{v}^2}{\varepsilon}+\frac{\overline{u}\overline{w}}{\alpha}\right) & -\alpha\varepsilon\lambda^4+\alpha\overline{v}\lambda^3+\left(\frac{\overline{w}}{2\alpha}-\frac{\alpha\overline{v}^2}{2\varepsilon}\right)\lambda^2+\left(\frac{\alpha\overline{v}^3}{2\varepsilon^2}-\frac{\overline{u}_x}{2\varepsilon}-\frac{\overline{u}^2}{2\varepsilon}\right)\lambda-\frac{1}{2\varepsilon^2}\overline{v}_{xx}+\frac{\overline{w}^2}{2\alpha\varepsilon}+\frac{3\overline{v}^2\overline{w}}{2\varepsilon^2}+\frac{\overline{u}^2\overline{v}}{\varepsilon^2}\\ \varepsilon\lambda^3+\overline{v}\lambda^2+\left(\frac{\overline{w}}{2\alpha}+\frac{\overline{v}^2}{2\varepsilon}\right)\lambda+\frac{1}{2\alpha\varepsilon}\left(2\overline{v}\,\overline{w}+\frac{\alpha\,\overline{v}^3}{\varepsilon}-\overline{u}_x+\overline{u}^2\right) & -\overline{u}\lambda^2-\frac{1}{2\varepsilon}\overline{v}_x\lambda-\frac{1}{2\varepsilon}\left(\frac{\overline{w}_x}{2\alpha}-\frac{\overline{u}\,\overline{v}^2}{\varepsilon}-\frac{\overline{u}\,\overline{w}}{\alpha}\right)\end{pmatrix}$$

与矩阵 $\overline{V}_2$ 具有相同形式，变换(16)，(17)和(18)式将原位势 u,v,w 映射为新位势 $\overline{u},\overline{v},\overline{w}$。于是得到下面的定理。

定理 1　由变换(16)，(17)和(18)式可从孤子方程(10)，(11)的一组解(u,v,w)生成它们的另一组新解($\overline{u}$，$\overline{v},\overline{w}$)，其中 a_0,b_0,c_0,d_0 由线性系统(20)唯一确定。称变换$(\phi,u,v,w)\to(\overline{\phi},\overline{u},\overline{v},\overline{w})$为孤子方程(10)，(11)的一个 Darboux 变换。

3 精确解

本节尝试利用 Darboux 变换来求孤子方程的精确解。以常数 $u=0,v=0,w=0$ 作为种子解，可以得到基本解。选取两个基本解为

$$\varphi(\lambda_j)=\begin{pmatrix}\sqrt{-\alpha\lambda_j}\cosh\xi_j\\ \sinh\xi_j\end{pmatrix},\psi(\lambda_j)=\begin{pmatrix}\sqrt{-\alpha\lambda_j}\sinh\xi_j\\ \cosh\xi_j\end{pmatrix},\tag{26}$$

其中，$\xi_j=h_j(x+\lambda_jy+\lambda_j^2t)$，$h_j=\varepsilon\lambda_j\sqrt{-\alpha\lambda_j}$，$j=1,2$。

根据 (20) 式有 $\sigma_j=-\frac{1}{\sqrt{-\alpha\lambda_j}}\frac{1-\gamma_j\tanh\xi_j}{\tanh\xi_j-\gamma_j}$，$j=1,2$。

由(19)和(21)式可知

$$\begin{cases}a_0=\frac{a_1(\lambda_1\sigma_2-\lambda_2\sigma_1)}{\sigma_1-\sigma_2}-\frac{\alpha a_1\sigma_1^2\sigma_2^2(\lambda_1-\lambda_2)^2}{(\sigma_1-\sigma_2)^2},\\ b_0=-\frac{a_1(\lambda_1-\lambda_2)}{\sigma_1-\sigma_2}+\frac{\alpha a_1\sigma_1\sigma_2(\lambda_1-\lambda_2)(\lambda_1\sigma_1-\lambda_2\sigma_2)}{(\sigma_1-\sigma_2)^2},\\ c_0=\frac{a_1\sigma_1\sigma_2(\lambda_1-\lambda_2)}{\sigma_1-\sigma_2},\\ d_0=-\frac{a_1(\lambda_1\sigma_1-\lambda_2\sigma_2)}{\sigma_1-\sigma_2},\end{cases}\tag{27}$$

令 $u=0,v=0,w=0$ 可得一组新解

$$v_1=v+\frac{\varepsilon}{a_1}(d_0-a_0)=\frac{\varepsilon}{a_1}(d_0-a_0),$$

$$u_1=u-\frac{c_0}{a_1}\alpha(v+\overline{v})+\frac{\varepsilon}{a_1}b_0=-\frac{\alpha\varepsilon}{a_1^2}c_0(d_0-a_0)+\frac{\varepsilon}{a_1}b_0,$$

$$w_1=\frac{2\alpha\varepsilon}{a_1}b_0c_0-\frac{\alpha\varepsilon}{a_1^2}(d_0^2-a_0^2)-\frac{2\alpha^2\varepsilon}{a_1^3}c_0^2(d_0-a_0)。\tag{28}$$

适当选取参数可以得到孤子图。

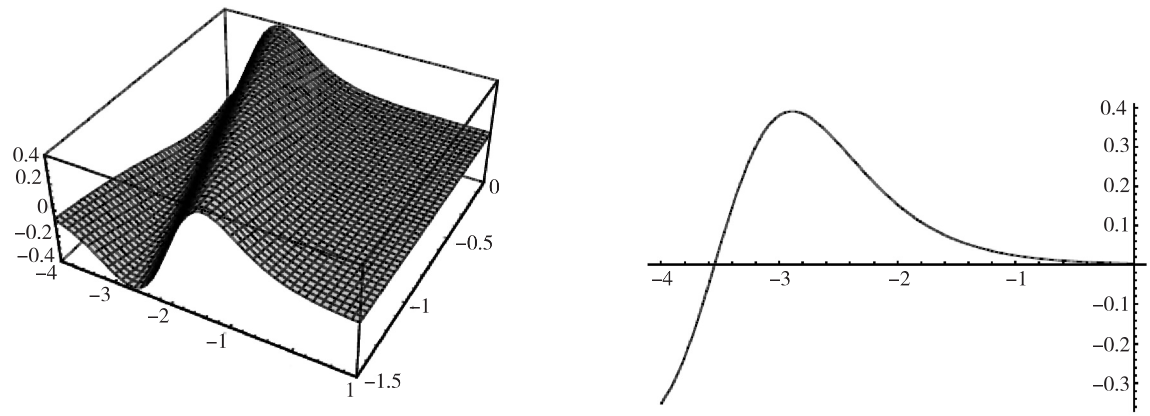

图 1u $\lambda_1=1,\lambda_2=1.01,\gamma_1=1,\gamma_2=-2,\alpha_1=-1,\varepsilon=1,a_1=1,t=0$

Fig. 1u The figures of the soliton solution

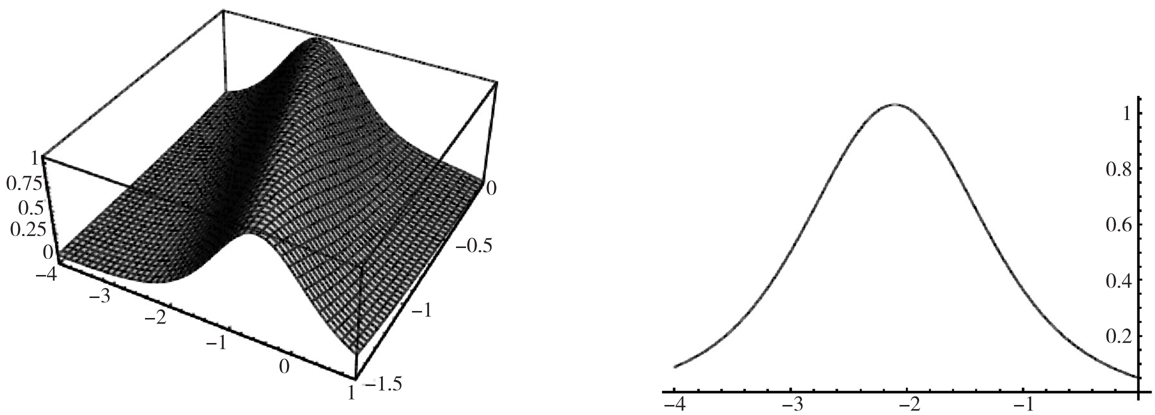

图 2v $\lambda_1=1,\lambda_2=1.01,\gamma_1=2,\gamma_2=3,\alpha_1=-1,\varepsilon=1,a_1=1,t=0$

Fig. 2v The figures of the soliton solution

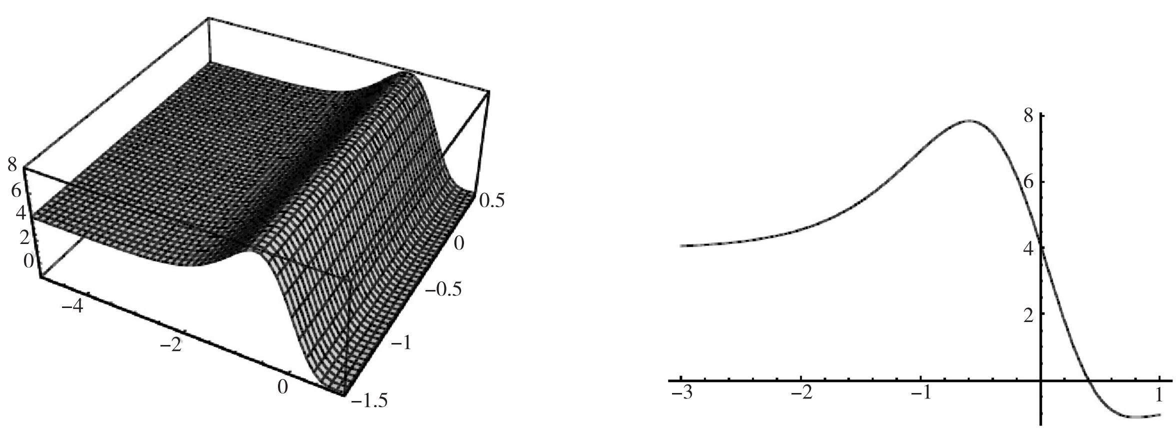

图 3w $\lambda_1=1,\lambda_2=2.2,\gamma_1=1,\gamma_2=1.25,\alpha_1=-1,\varepsilon=1,a_1=1$

Fig. 3w The figures of the soliton solution

以 u_1,v_1,w_1 为种子解,作迭代选取两组基本解

$$\begin{cases}\bar{\phi}(\lambda_j)=\begin{pmatrix}\bar{\phi}_1(\lambda_j)\\ \bar{\phi}_2(\lambda_j)\end{pmatrix}=\begin{pmatrix}a_1\lambda_j+a_0 & -\alpha c_0\lambda_j+b_0\\ c_0 & a_1\lambda_j+d_0\end{pmatrix}\begin{pmatrix}\sqrt{-\alpha\lambda_j}\cosh\xi_j\\ \sinh\xi_j\end{pmatrix}=\\ \qquad\begin{pmatrix}(a_1\lambda_j+a_0)\sqrt{-\alpha\lambda_j}\cosh\xi_j+(-\alpha c_0\lambda_j+b_0)\sinh\xi_j\\ c_0\sqrt{-\alpha\lambda_j}\cosh\xi_j+(a_1\lambda_j+d_0)\sinh\xi_j\end{pmatrix},\\ \bar{\psi}(\lambda_j)=\begin{pmatrix}\bar{\psi}_1(\lambda_j)\\ \bar{\psi}_2(\lambda_j)\end{pmatrix}=\begin{pmatrix}a_1\lambda_j+a_0 & -\alpha c_0\lambda_j+b_0\\ c_0 & a_1\lambda_j+d_0\end{pmatrix}\begin{pmatrix}\sqrt{-\alpha\lambda_j}\sinh\xi_j\\ \cosh\xi_j\end{pmatrix}=\\ \qquad\begin{pmatrix}(a_1\lambda_j+a_0)\sqrt{-\alpha\lambda_j}\sinh\xi_j+(-\alpha c_0\lambda_j+b_0)\cosh\xi_j\\ c_0\sqrt{-\alpha\lambda_j}\sinh\xi_j+(a_1\lambda_j+d_0)\cosh\xi_j\end{pmatrix},\end{cases}\tag{29}$$

其中，$\xi_j=h_j(x+\lambda_j y+\lambda_j^2 t)$，$h_j=\varepsilon\lambda_j\sqrt{-\alpha\lambda_j}$，$j=1,2$。

根据 (20) 式有

$$\bar{\sigma}_j=\frac{\bar{\varphi}_2(\lambda_j)-\bar{\gamma}_j\bar{\psi}_2(\lambda_j)}{\bar{\varphi}_1(\lambda_j)-\bar{\gamma}_j\bar{\psi}_1(\lambda_j)}=\frac{c_0\sqrt{-\alpha\lambda_j}(\cosh\xi_j-\bar{\gamma}_j\sinh\xi_j)+(\lambda_j a_1+d_0)(\sinh\xi_j-\bar{\gamma}_j\cosh\xi_j)}{(\lambda_j a_1+a_0)\sqrt{-\alpha\lambda_j}(\cosh\xi_j-\bar{\gamma}_j\sinh\xi_j)+(-\alpha c_0\lambda_j+b_0)(\sinh\xi_j-\bar{\gamma}_j\cosh\xi_j)},$$

代入

$$\begin{cases}\bar{a}_0+a_1\lambda_j+(\bar{b}_0-\alpha\bar{c}_0\lambda_j)\bar{\sigma}_j=0,\\ \bar{c}_0+(a_1\lambda_j+\bar{d}_0)\bar{\sigma}_j=0,\end{cases}$$

解

$$\begin{cases}\bar{a}_0=\dfrac{a_1(\lambda_1\bar{\sigma}_2-\lambda_2\bar{\sigma}_1)}{\bar{\sigma}_1-\bar{\sigma}_2}-\dfrac{\alpha a_1\bar{\sigma}_1^2\bar{\sigma}_2^2(\lambda_1-\lambda_2)^2}{(\bar{\sigma}_1-\bar{\sigma}_2)^2},\\ \bar{b}_0=-\dfrac{a_1(\lambda_1-\lambda_2)}{\bar{\sigma}_1-\bar{\sigma}_2}+\dfrac{\alpha a_1\bar{\sigma}_1\bar{\sigma}_2(\lambda_1-\lambda_2)(\lambda_1\bar{\sigma}_1-\lambda_2\bar{\sigma}_2)}{(\bar{\sigma}_1-\bar{\sigma}_2)^2},\\ \bar{c}_0=\dfrac{a_1\bar{\sigma}_1\bar{\sigma}_2(\lambda_1-\lambda_2)}{\bar{\sigma}_1-\bar{\sigma}_2},\\ \bar{d}_0=-\dfrac{a_1(\lambda_1\bar{\sigma}_1-\lambda_2\bar{\sigma}_2)}{\bar{\sigma}_1-\bar{\sigma}_2},\end{cases}\tag{30}$$

可得另一组新解

$$\begin{cases}v_2=v_1+\dfrac{\varepsilon}{a_1}(\bar{d}_0-\bar{a}_0),\\ u_2=u_1-\dfrac{\alpha\bar{c}_0}{a_1}(v_1+v_2)+\dfrac{\varepsilon}{a_1}\bar{b}_0,\\ w_2=w_1-\dfrac{2\alpha}{a_1}\bar{c}_{0,x}-\dfrac{\alpha\varepsilon}{a_1^2}(\bar{d}_0^2-\bar{a}_0^2)-\dfrac{2\alpha}{a_1}(\bar{d}_0-\bar{a}_0)v_2。\end{cases}\tag{31}$$

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Darboux Transformation of a New Soliton Equations and Its Exact Solutions

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Abstract: Starting from a new 2×2 matrix spectral problem, we propose a nonlinear soliton equation. For the first two non-trivial soliton equations, gauge transformation of the Lax pair of the equation is found by using the fundamental solution matrix of the matrix spectral problem. Then, the Darboux transformation of the soliton equation is obtained. As an application, some exact solutions of the equation are given, and several interesting figures of the solutions are plotted, it helps to analysis and research the natural phenomena described by the equations.

Key words: zero curvature equation; Darboux transformation; gauge transformation; exact solution

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