

一类分数阶 BAM 神经网络的 Mittag-Leffler 稳定性*

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摘要:利用解的唯一性定理以及分数阶导数的相关性质,研究了分数阶 BAM 神经网络的 Mittag-Leffler 稳定性,并给出了相应的充分性条件,最后通过实例仿真验证了结论的正确性。

关键词:分数阶;神经网络;Mittag-Leffler;稳定

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自从 1987 年, Kosko 提出了 BAM 神经网络模型^[1]:

$$\begin{cases} \frac{dx_i}{dt} = -c_i x_i + \sum_{j=1}^q m_{ij} f_j(y_j) + I_i, & i = 1, 2, \dots, p \\ \frac{dy_j}{dt} = -b_j y_j + \sum_{i=1}^p n_{ij} g_i(x_i) + J_j, & j = 1, 2, \dots, q \end{cases}, \quad (1)$$

对于 BAM 神经网络模型的学术研究就日渐加深,其中时滞 BAM 神经网络模型^[2-3]、随机 BAM 神经网络模型^[4-5]和周期 BAM 神经网络模型^[6]是近几年来在这方面的研究热点。另外,随着分数阶微分方程与稳定性理论的逐渐成熟,分数阶神经网络模型^[7]在 2009 年被 Arefeh Boroomand, Mohammad 和 Menha 通过利用广义的电容器去替换一般电容器的方法提出来:

$$F_i \frac{d^\alpha x_i}{dt^\alpha} = -\frac{x_i}{R_i} + \sum_{j=1}^n T_{ij} g_j(x_j) + I_i, \quad i = 1, 2, \dots, n, \quad 0 < \alpha < 1, \quad (2)$$

其中, $\frac{d^\alpha x_i}{dt^\alpha}$ 为 Caputo 型分数阶导数。在此基础上,本文给出一类分数阶 BAM 神经网络模型:

$$\begin{cases} \frac{d^\alpha x_i}{dt^\alpha} = -c_i x_i + \sum_{j=1}^n m_{ij} f_j(y_j) + I_i \\ \frac{d^\alpha y_i}{dt^\alpha} = -b_i y_i + \sum_{j=1}^n n_{ij} g_j(x_j) + J_i \end{cases}, \quad i = 1, 2, \dots, n. \quad (3)$$

进而讨论了该模型稳定性充分条件。

1 预备知识

定义 1^[8] 对于 $\alpha \in \mathbf{R}$, 函数 $f(x)$ 的分数阶 Caputo 导数 $D^\alpha f(x)$ 定义为:

$$D^\alpha f(x) = J^{m-\alpha} f^{(m)}(x), \quad (4)$$

其中, $m = [\alpha] + 1$, $J^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t) dt$, $\beta > 0$, 这里的 $\Gamma(\cdot)$ 为 Γ -函数, 即 $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt$ 。

性质 1 $D^\alpha C = 0$, 这里 C 为常数。

性质 2 $D^\alpha(\mu f(t) + \nu g(t)) = \mu D^\alpha f(t) + \nu D^\alpha g(t)$, 其中 μ, ν 为常数。

定义 2^[9] Mittag-Leffler 函数 $E_\alpha(z)$ 与双参数形式的 Mittag-Leffler 函数分别定义如下:

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$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \alpha > 0, z \in C, \quad (5)$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \alpha > 0, \beta > 0, z \in C. \quad (6)$$

由定义得, $E_\alpha(z) = E_{\alpha,1}(z)$, 并且 $E_{1,1}(z) = e^z$ 。

考虑一般的分数阶系统:

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = f(t, x(t)) \\ x(t_0) = x_{t_0} \end{cases}, 0 < \alpha < 1. \quad (7)$$

定义 3 若 $\bar{x} = 0$ 是系统(7)的平衡点, 且 $\|x(t)\| \leq [m(x_{t_0})E_\alpha(-\lambda(t-t_0)^\alpha)]^b$, 其中 $\lambda > 0, b > 0, m(0) = 0, m(x) \geq 0, |m(x_2) - m(x_1)| \leq m_0 |x_2 - x_1|$, 则称系统(7)的解是 Mittag-Leffler 稳定的。

引理 1^[10] 若系统(7)中 $f(t, x)$ 关于变量 x 满足 Lipschitz 条件, 则系统(7)存在唯一解。

引理 2^[11] 若 $V(t)$ 为 $[0, +\infty)$ 上的连续函数, 且满足 $D^\alpha V(t) \leq \theta V(t)$, 其中 $0 < \alpha < 1, \theta$ 为常数, 则 $V(t) \leq V(0)E_\alpha(\theta t^\alpha), t \geq 0$ 。

2 主要结果

考虑如下形式的分数阶 BAM 神经网络模型:

$$\begin{cases} \frac{d^\alpha \mathbf{x}}{dt^\alpha} = -\mathbf{C}\mathbf{x} + \mathbf{M}\mathbf{F}(\mathbf{y}) + \mathbf{I} \\ \frac{d^\alpha \mathbf{y}}{dt^\alpha} = -\mathbf{B}\mathbf{y} + \mathbf{N}\mathbf{G}(\mathbf{x}) + \mathbf{J} \end{cases}, 0 < \alpha < 1, \quad (8)$$

其中, $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top, \mathbf{y} = (y_1, y_2, \dots, y_n)^\top$ 表示神经元的状态变量; $\mathbf{C} = \text{diag}[c_1, c_2, \dots, c_n]^\top, \mathbf{B} = \text{diag}[b_1, b_2, \dots, b_n]^\top; c_i, b_j > 0$ 表示神经元自反馈; \mathbf{M}, \mathbf{N} 是神经元间的互联矩阵, $\mathbf{I} = (I_1, I_2, \dots, I_n)^\top, \mathbf{J} = (J_1, J_2, \dots, J_n)^\top$ 为表示偏置项的常数向量; $\mathbf{F}(\mathbf{x}) = [f_1(x_1), f_2(x_2), \dots, f_n(x_n)]^\top, \mathbf{G}(\mathbf{x}) = [g_1(x_1), g_2(x_2), \dots, g_n(x_n)]^\top$ 为激活函数, 且 $f_i(0) = 0, g_i(0) = 0$ 。

设 $(\mathbf{x}_e, \mathbf{y}_e)$ 是(8)式的平衡点, 令 $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_e, \tilde{\mathbf{y}} = \mathbf{y} - \mathbf{y}_e$, 则通过平移变换后可得到相应的平移系统:

$$\begin{cases} \frac{d^\alpha \tilde{\mathbf{x}}}{dt^\alpha} = -\mathbf{C}\tilde{\mathbf{x}} + \mathbf{M}\mathbf{F}(\tilde{\mathbf{y}}) \\ \frac{d^\alpha \tilde{\mathbf{y}}}{dt^\alpha} = -\mathbf{B}\tilde{\mathbf{y}} + \mathbf{N}\mathbf{G}(\tilde{\mathbf{x}}) \end{cases}, 0 < \alpha < 1, \quad (9)$$

这里 $\mathbf{F}(\tilde{\mathbf{y}}) = \mathbf{F}(\mathbf{y}) - \mathbf{F}(\mathbf{y}_e), \mathbf{G}(\tilde{\mathbf{x}}) = \mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{x}_e)$, 为了不引入新的符号, 将系统(9)内的“ \sim ”省略, 从而系统(9)可改写为:

$$\begin{cases} \frac{d^\alpha \mathbf{x}}{dt^\alpha} = -\mathbf{C}\mathbf{x} + \mathbf{M}\mathbf{F}(\mathbf{y}) \\ \frac{d^\alpha \mathbf{y}}{dt^\alpha} = -\mathbf{B}\mathbf{y} + \mathbf{N}\mathbf{G}(\mathbf{x}) \end{cases}, 0 < \alpha < 1, \quad (10)$$

要证明系统(8)中平衡点 $(\mathbf{x}_e, \mathbf{y}_e)$ 的 Mittag-Leffler 稳定性, 只需证明系统中平衡点 $(x, y) = (0, 0)$ 的 Mittag-Leffler 稳定性即可。

定理 若系统(10)满足:

(H₁) $f_i(t), g_j(t)$ 分别满足 Lipschitz 常数为 u_i, v_i 的 Lipschitz 条件;

(H₂) $\lambda > \mu$, 其中 $\lambda = \min\{c_i, b_i\}, \mu = \max\{\sum_{j=1}^n |m_{ji}|u_i, \sum_{j=1}^n |n_{ji}|v_i\}$, 则 $(x, y) = (0, 0)$ 是系统(10)的 Mittag-Leffler 稳定平衡点。

证明 假设 $(\mathbf{x}, \mathbf{y}) \neq (0, 0)$ 为系统(10)的解, 由引理 1 得, $x_i(t)x_i(0) > 0, y_i(t)y_i(0) > 0, t \geq 0$ 。若 $x_i(0) > 0$, 则 $x_i(t) > 0$, 从而有 $D^\alpha |x_i(t)| = D^\alpha x_i(t)$; 若 $x_i(0) < 0$, 则 $x_i(t) < 0$, 从而有 $D^\alpha |x_i(t)| = D^\alpha (-x_i(t)) = -D^\alpha x_i(t)$;

故 $D^\alpha |x_i(t)| = \text{sgn}(x_i(t))D^\alpha x_i(t)$ 。同理可得, $D^\alpha |y_i(t)| = \text{sgn}(y_i(t))D^\alpha y_i(t)$ 。

构造下列两个辅助函数, $V_1(t) = \sum_{i=1}^n |x_i(t)|, V_2(t) = \sum_{j=1}^n |y_j(t)|$, 则由假设 $(H_1), (H_2)$ 得:

$$\begin{aligned} D^\alpha V_1(t) &= D^\alpha \left(\sum_{i=1}^n |x_i(t)| \right) = \sum_{i=1}^n D^\alpha |x_i(t)| = \sum_{i=1}^n \text{sgn}(x_i(t))D^\alpha x_i(t) = \\ & \sum_{i=1}^n \text{sgn}(x_i(t))(-c_i x_i(t) + \sum_{j=1}^n m_{ij} f_j(y_j(t))) \leq \sum_{i=1}^n (-c_i |x_i(t)| + \sum_{j=1}^n |m_{ij}| |u_j| |y_j(t)|) = \\ & - \sum_{i=1}^n c_i |x_i(t)| + \sum_{i=1}^n \sum_{j=1}^n |m_{ij}| |u_j| |y_j(t)| = - \sum_{i=1}^n c_i |x_i(t)| + \sum_{i=1}^n \sum_{j=1}^n |m_{ji}| |u_i| |y_i(t)|. \end{aligned} \quad (10)$$

类似地,

$$\begin{aligned} D^\alpha V_2(t) &= D^\alpha \left(\sum_{j=1}^n |y_j(t)| \right) = \sum_{j=1}^n D^\alpha |y_j(t)| = \sum_{j=1}^n \text{sgn}(y_j(t))D^\alpha y_j(t) = \\ & \sum_{j=1}^n \text{sgn}(y_j(t))(-b_j y_j(t) + \sum_{i=1}^n n_{ij} g_i(x_i(t))) \leq \sum_{j=1}^n (-b_j |y_j(t)| + \sum_{i=1}^n |n_{ij}| |v_i| |x_i(t)|) = \\ & - \sum_{j=1}^n b_j |y_j(t)| + \sum_{i=1}^n \sum_{j=1}^n |n_{ij}| |v_j| |x_j(t)| = - \sum_{i=1}^n b_i |y_i(t)| + \sum_{i=1}^n \sum_{j=1}^n |n_{ji}| |v_i| |x_i(t)|, \end{aligned} \quad (12)$$

从而

$$\begin{aligned} D^\alpha (V_1(t) + V_2(t)) &\leq \sum_{i=1}^n -c_i |x_i(t)| + \sum_{i=1}^n \sum_{j=1}^n |m_{ji}| |u_i| |y_i(t)| + \sum_{i=1}^n -b_i |y_i(t)| + \sum_{i=1}^n \sum_{j=1}^n |n_{ji}| |v_i| |x_i(t)| \leq \\ & -\lambda \sum_{i=1}^n (|x_i(t)| + |y_i(t)|) + \mu \sum_{i=1}^n (|x_i(t)| + |y_i(t)|) = (-\lambda + \mu) \sum_{i=1}^n (|x_i(t)| + |y_i(t)|) = \\ & (-\lambda + \mu)(V_1(t) + V_2(t)). \end{aligned} \quad (11)$$

从而由引理 2 得, $V_1(t) + V_2(t) \leq (V_1(0) + V_2(0))E_\alpha((-\lambda + \mu)t^\alpha)$, 即

$$\sum_{i=1}^n (|x_i(t)| + |y_i(t)|) \leq \left(\sum_{i=1}^n (|x_i(0)| + |y_i(0)|) \right) E_\alpha((-\lambda + \mu)t^\alpha). \quad (12)$$

所以, $(\mathbf{x}, \mathbf{y}) = (0, 0)$ 是系统(10) Mittag-Leffler 稳定的平衡点。

证毕

3 实例仿真

为系统(10)选取如下参数:

$$\alpha = 0.8, \mathbf{C} = \begin{pmatrix} 2.1 & 0 \\ 0 & 1.7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1.8 & 0 \\ 0 & 2.4 \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 0.3 & 0.4 \\ 0.2 & 0.4 \end{pmatrix}, \mathbf{N} = \begin{pmatrix} 0.1 & 0.3 \\ 0.4 & 0.2 \end{pmatrix}.$$

选取 $\mathbf{F}(\mathbf{y}) = [\arctan y_1, \arctan y_2]^\top, \mathbf{G}(\mathbf{x}) = [\arctan x_1, \arctan x_2]^\top$ 则系统(10)满足定理条件, 因此该系统的平衡点 $(\mathbf{x}, \mathbf{y}) = (0, 0)$ 是 Mittag-Leffler 稳定的。图 1~图 4 分别为 $x_1(t), x_2(t), y_1(t), y_2(t)$ 随时间 t 收敛于 0 的曲线图。

由仿真结果可以看出, 系统(10)的状态曲线收敛于平衡点 $(x, y) = (0, 0)$ 。

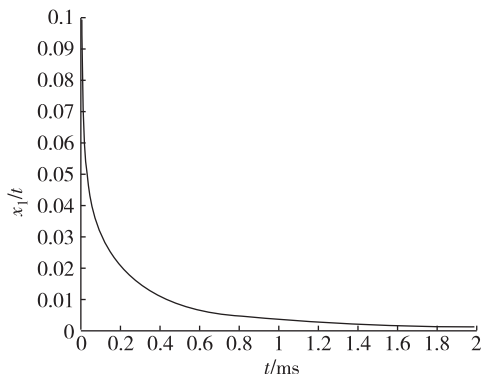


图 1 $x_1(t)$ 的状态曲线图

Fig. 1 The curve of $x_1(t)$

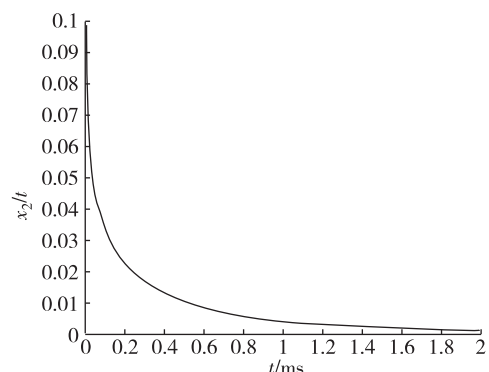
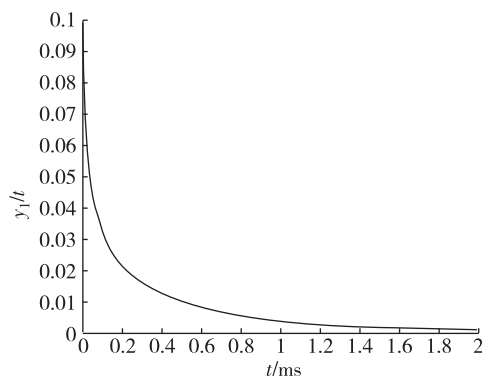
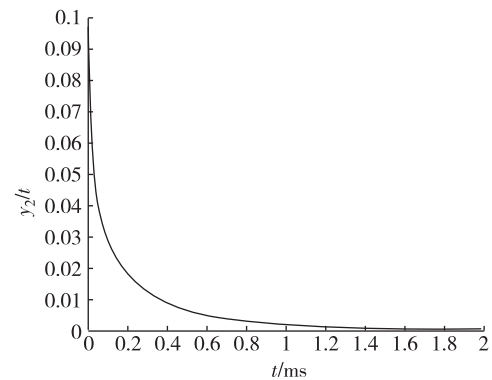


图 2 $x_2(t)$ 的状态曲线图

Fig. 2 The curve of $x_2(t)$

图3 $y_1(t)$ 的状态曲线图Fig. 3 The curve of $y_1(t)$ 图4 $y_2(t)$ 的状态曲线图Fig. 4 The curve of $y_2(t)$

4 结语

通过对分数阶 BAM 神经网络的性态分析,得到了如何判定 BAM 神经网络系统稳定性的充分性条件,进而对所得结果进行了 Matlab 仿真,通过实例验证了文中结论的正确性。

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Mittag-Leffler Stability Analysis of a Class of Fractional-order BAM Neural Networks

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Abstract: Making use of the property of fractional order derivative and the uniqueness of the solution, the Mittag-Leffler stability of fractional order BAM neural networks was investigated, and a sufficient condition was gotten. At last, the correctness of the conclusion was verified by an emulating example.

Key words: fractional order; neural networks; Mittag-Leffler; stability

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