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Modified Shifted Laplace Preconditioners for Symmetric Complex Linear Systems*

HE Jun, LIU Yanmin, TIAN Junkang, ZHANG Zhuanzhou

(School of Mathematics and Computational Science, Zunyi Normal College, Zunyi Guizhou 563002, China)

Abstract: Modified shifted Laplace preconditioners are introduced to the solution of complex linear systems, which are frequently indefinite and large, it is difficult to solve iteratively. The spectral characteristics of the preconditioners are studied, and we find that, all eigenvalues of the preconditioned matrices are located on one circle and strongly clustered. Numerical experiments on the Helmholtz equations are presented to illustrate the numerical effectiveness of our preconditioners.

Key words: modified shifted Laplace preconditioner; complex linear system; Krylov method

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1 Introduction

Consider the nonsingular complex linear systems as follows:

$$\boldsymbol{C}x=b, \text{with } \boldsymbol{C}=\boldsymbol{A}+\mathrm{i}\boldsymbol{B}, \tag{1}$$

Where $\mathrm{i}=\sqrt{-1}$, $\boldsymbol{A},\boldsymbol{B}\in\mathbf{R}^{n\times n}$, $\boldsymbol{A}^{\mathrm{T}}=\boldsymbol{A}$, $\boldsymbol{B}^{\mathrm{T}}=\boldsymbol{B}$. Then, we can find that the matrix $\boldsymbol{C}$ is symmetric, but non-Hermitian. Complex systems arise in a number of applications, such as time-harmonic wave propagations, scattering phenomena which is arisen in optical and acoustic problems. In our numerical experiments, we use the examples of the Helmholtz equation which is discretized by finite difference.

The coefficient matrices are large and usually sparse, Krylov subspace methods are better choice when a

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The first author biography: He Jun, male, associate professor, Doctor's degree, mainly engaged in numerical analysis and matrix theory, E-mail: hejunfan1@163.com

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作者简介:何军,男,副教授,博士,研究方向为数值分析及矩阵理论,E-mail:hejunfan1@163.com

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good preconditioner is introduced. In this paper, modified shifted Laplace preconditioners are introduced to improve the convergence of the Krylov subspace methods.

Some effective preconditioners for complex linear systems are introduced in [1-4]. Axelsson, Kucherov[1] and Benzi[4] presented some effective preconditioners based the $2n \times 2n$ equivalent real formulations instead of the $n \times n$ complex systems.

Laird[2] took the Laplace operator perturbed by a real-valued linear term as a preconditioner and improved the convergence rate of iterative methods. In [5-8], the class of shifted Laplace preconditioners are further generalized by also considering general complex shifts. The spectral characteristics of the shifted Laplace preconditioners are studied by Van Gijzen, Erlangga, Vuik in [8], and we can find that these preconditioners are effective and simple to construct.

Recently, based on the Hermitian and skew-Hermitian parts of the coefficient matrix $\boldsymbol{C}$, when $\boldsymbol{A}$ is a symmetric positive semidefinite matrix and $\boldsymbol{B}$ is a symmetric positive definite matrix, or $\boldsymbol{A}$ is a symmetric positive definite matrix and $\boldsymbol{B}$ is a symmetric positive semidefinite matrix, Bai et al. proposed in [13] a modified Hermitian and skew-Hermitian splitting (MHSS) iteration method, which is more efficient than the HSS iteration method[14] for solving the complex system (1). It has been proved that the MHSS iteration method is convergent unconditionally in [13]. In[15-17], the authors gave some variants of MHSS and established the convergence theories for the these iteration method under suitable conditions.

This paper gives the class of preconditioners just from an algebraic point of view, and is based on the work of [5-8]. To distinguish it from the shifted Laplace preconditioners, we refer them as modified shifted Laplace preconditioners.

In Section 2 of this paper, we will establish the modified shifted Laplace preconditioners and study the spectral properties of the preconditioners for the complex linear systems. In Section 3 of this paper, we give some numerical experiments on the Helmholtz equations. Finally, conclusions are obtained in Section 4.

2 The modified shifted Laplace preconditioners

In this paper, we consider the following preconditioned linear system:

$$P^{-1}\boldsymbol{C}x = P^{-1}b, \tag{2}$$

where the preconditioner P is non-singular. The preconditioner is very important. Generally speaking, we should choose the preconditioner P to let the condition number of the preconditioned matrix $P^{-1}\boldsymbol{C}$ is less than the condition number of the original matrix $\boldsymbol{A}$, you can turn to [4] for more details about the preconditioner.

In [8], Van Gijzen, Erlangga, Vuik introduced a shifted Laplace preconditioner:

$$\boldsymbol{L} + \mathrm{i}\boldsymbol{C} - z_2\boldsymbol{M}$$

to precondition the Helmholtz operator:

$$\boldsymbol{L} + \mathrm{i}\boldsymbol{C} - z_1\boldsymbol{M}.$$

When we turn to the complex linear systems of the form (1), we can rewrite the coefficient matrix as follows:

$$\boldsymbol{C} = \boldsymbol{A} + \alpha\boldsymbol{B} - (\alpha - \mathrm{i})\boldsymbol{B} = \boldsymbol{L} - z_1\boldsymbol{B}, \tag{3}$$

where $\boldsymbol{L} = \boldsymbol{A} + \alpha\boldsymbol{B}$, $z_1 = \alpha - \mathrm{i} = \alpha_1 + \mathrm{i}\beta_2$, in this paper, we all choose $\alpha > 0$. Looking for a form of $\boldsymbol{C}$, we consider the preconditioner:

$$\boldsymbol{P} = \boldsymbol{L} - z_2\boldsymbol{B}, \tag{4}$$

where $z_2 = \alpha_2 + \mathrm{i}\beta_2$, in order to distinguish it from the shifted Laplace preconditioners for Helmholtz operator, we refer them as modified shifted Laplace preconditioners. We can find that these preconditioners are still easy to construct.

In [12], for the complex linear system:

$$(\boldsymbol{A}-h^2 pI+\mathrm{i}h^2 qI)x=b,$$

if we choose $\alpha_2=\frac{\alpha}{q}$, $\beta=-1$, then, we can get the preconditioner:

$$P_1=\boldsymbol{A}+h^2(\alpha-p)I+\mathrm{i}h^2 qI(\alpha>p),$$

which is introduced in [12]. So, we can consider the modified shifted Laplace preconditioners as a generation of the preconditioner in [12]. In our numerical experiments, we will compare the results with the results of the modified shifted Laplace preconditioners.

In the following Theorem, we will discuss the spectrum results of the preconditioned matrix $P^{-1}\boldsymbol{C}$ when $\boldsymbol{B}$ is a symmetric, real and nonsingular matrix, the proof of the Theorem is similar to the proof in [8].

Theorem 1 Let $\beta_2\neq 0$, and assume that $\boldsymbol{L}$ is a symmetric real matrix, $\boldsymbol{B}$ is a symmetric, real and nonsingular matrix. Then, the eigenvalues $\lambda=\lambda^r+\mathrm{i}\lambda^i$ of $P^{-1}\boldsymbol{A}$ are located on the circle given by

$$\left(\lambda^r-\frac{\beta_2+\beta_1}{2\beta_2}\right)^2+\left(\lambda^i-\frac{\alpha_2-\alpha_1}{2\beta_2}\right)^2=\frac{(\beta_2-\beta_1)^2+(\alpha_2-\alpha_1)^2}{(2\beta_2)^2}, \tag{5}$$

the center c of this circle is

$$c=\left(\frac{\beta_2+\beta_1}{2\beta_2},\frac{\alpha_2+\alpha_1}{2\beta_2}\right), \tag{6}$$

and the radius R is

$$R=\sqrt{\frac{(\beta_2-\beta_1)^2+(\alpha_2-\alpha_1)^2}{(2\beta_2)^2}}. \tag{7}$$

Proof Let $\lambda=\lambda^r+\mathrm{i}\lambda^i$ be an eigenvalue of $P^{-1}\boldsymbol{A}$ with eigenvector x. $\boldsymbol{B}$ is a symmetric, real and nonsingular matrix, then we can get

$$(\boldsymbol{L}-z_1\boldsymbol{B})x=\lambda(\boldsymbol{L}-z_2\boldsymbol{B})x, \tag{8}$$

and it is easy to see that $\boldsymbol{L}-z_1\boldsymbol{B}$ and $\boldsymbol{L}-z_2\boldsymbol{B}$ share the same eigenvectors, which are the eigenvectors of

$$\boldsymbol{L}x=\omega\boldsymbol{B}x.$$

Substituting this into (8) yields

$$\omega-z_1=\lambda(\omega-z_2), \tag{9}$$

and we know that, $\lambda=\lambda^r+\mathrm{i}\lambda^i$, $z_1=\alpha_1+\mathrm{i}\beta_1$, $z_2=\alpha_2+\mathrm{i}\beta_2$, substituting this into (9) yields

$$\begin{cases}\omega-\alpha_1=\lambda^r(\omega-\alpha_2)+\lambda^i\beta_2\\ -\beta_1=-\lambda^r\beta_2+\lambda^i(\omega-\alpha_2)\end{cases}. \tag{10}$$

Then, by direct computation, we can get

$$\left(\lambda^r-\frac{\beta_2+\beta_1}{2\beta_2}\right)^2+\left(\lambda^i-\frac{\alpha_2-\alpha_1}{2\beta_2}\right)^2=\frac{(\beta_2-\beta_1)^2+(\alpha_2-\alpha_1)^2}{(2\beta_2)^2}.$$

The center of the circle can also be written as

$$c=\frac{z_1-\overline{z_2}}{z_2-\overline{z_2}},$$

and the radius as

$$R=\left|\frac{z_2-z_1}{z_2-\overline{z_2}}\right|.$$

Theorem 2 Let $\beta_2=0$, and assume that $\boldsymbol{L}$ is a symmetric real matrix, $\boldsymbol{B}$ is a symmetric, real and nonsingular matrix. Then, the eigenvalues $\lambda=\lambda^r+\mathrm{i}\lambda^i$ of $P^{-1}\boldsymbol{A}$ are located on the circle given by

$$-\beta_1\lambda^r+(\alpha_1-\alpha_2)\lambda^i+\beta_1=0$$

Proof Let $\beta_2=0$, from (10) and then by direct computation, we can get

$$-\beta_1\lambda^r+(\alpha_1-\alpha_2)\lambda^i+\beta_1=0.$$

Remark If the complex linear system is nonsymmetric or $\boldsymbol{B}$ is singular, the modified shifted Laplace preconditioner can still be applied to this case, but we can't get the similar spectrum results of the preconditioned matrix, see example 2.

3 Some considerations

In [8], the authors give the choice of the optimal shift, that is:

$$z_2 = -\mathrm{i}|z_1|,$$

in our test, we also give the choice. Another choice is the constant α, if $\alpha \to 0$, the preconditioner P corresponds to using the original operator $\boldsymbol{C}$, which means that performing the preconditioning operation is as hard as solving the original system. So, we should choose a suitable α. In our numerical experiments, we all take $\alpha=1$.

The basic algorithm of Krylov subspace methods is the conjugate gradient method (CG), only three vectors is used in memory. When the coefficient matrix $\boldsymbol{C}$ is symmetric, and positive definite, the conjugate gradient method work well. Otherwise, the conjugate gradient method may break down. GMRES has the advantage that theoretically the algorithm does not break down unless convergence has been reached. The main problem in GMRES is that, when the iteration number increases, the amount of storage also increases. Therefore, the application of GMRES may be limited by the computer storage. To remedy this problem, a restarted version, GMRES(m). In our numerical experiments, we all take the restarted GMRES(m).

4 Numerical examples

Helmholtz equations are very important in the wave propagation phenomena. Similar to the numerical experiments in [4], based on the finite-difference discretization of the partial differential equation (PDE), two types of model problems were considered:

$$-\Delta u - pu + \mathrm{i}qu = f. \tag{11}$$

All the numerical experiments were performed with Matlab 7.0. In all of our runs we used a zero initial guess and let $r^{(k)}$ to be the residual vector after $k-th$ iteration. When $\| r^{(k)} \|_2 / \| r^{(0)} \|_2 \leqslant 10^{-6}$, The algorithm stop. The right-hand side vectors $b=\boldsymbol{Ce}$, $\boldsymbol{e}=(1,1,\cdots,1)^{\mathrm{T}}$. We will use preconditioned GMRES(10) to solve the complex linear systems.

Example 1 We consider (11) on the $2D$ domain $\Omega=[0,1]\times[0,1]$ with different values of p, q and boundary conditions, and we let $p>0$, $q>0$.

We discretize these problems by finite differences using uniform $m\times m$ grids, then, we can get the following complex linear system which is $n\times n$ with $n=m^2$.

$$\boldsymbol{C}x=(T-h^2pI+\mathrm{i}h^2\boldsymbol{D})x=b, \tag{12}$$

when $p=200$, the diagonal matrix $\boldsymbol{D}$ has random entries in the interval $(0,200)$. We can find that, $\boldsymbol{A}$ is indefinite and $\boldsymbol{B}$ is positive definite. In this example, we change the preconditioner which is introduced in [12] to:

$$P_1=\boldsymbol{A}+h^2(\alpha-p)I+\mathrm{i}h^2q\boldsymbol{D}, \alpha=200,$$

and give the modified shifted Laplace preconditioner:

$$P_2=\boldsymbol{A}-h^2pI+h^2q\boldsymbol{D}+\mathrm{i}|1-\mathrm{i}|h^2q\boldsymbol{D}.$$

From fig. 1, we can get the results that the eigenvalues of $P^{-1}\boldsymbol{A}$ are strongly clustered, and are located in a circle, which are expected in Theorem 1.

From fig. 2, we can find that the modified shifted Laplace preconditioners perform well, and are more effective than the preconditioners which are introduced in [12].

From Tab. 1, we can see the iterations counts for different grids, we find that the convergence behavior of this preconditioner is independent of the gridsize h.

Tab. 1 GMRES iterations for different grids for example 1

Grid	No preconditioning	$P=P1$	$P=P2$
10×10	55	27	14
20×20	114	27	14
30×30	186	29	15
40×40	275	28	14

Example 2 In this example, we let $p=-200, q=0$ and a boundary condition of the form

$$\frac{\partial u}{\partial n}+\beta u=0 \text{ on } \{(1,y)\,|\,0<y<1\},$$

and Dirichlet boundary conditions are satisfied on the remaining three sides of the unit square, with the complex function β chosen so as to make the imaginary part of $\boldsymbol{C}$ indefinite. Taking suitable β, we can get the complex linear system of the form (12), where the diagonal entries of $\boldsymbol{D}$ is list as follows:

$$d_{i+(j-1)m}=\begin{cases}(-1)^{j-1}100/h, & \text{when } i=m\\ 0, & \text{else}\end{cases},$$

$i,j=1,\cdots,m$. Using the five point finite difference to discretize the Helmholtz equation (11) results in the following linear system:

$$\boldsymbol{C}x=(\boldsymbol{T}-h^2 pI+\mathrm{i}\boldsymbol{D})x=(\boldsymbol{A}+\mathrm{i}\boldsymbol{B})x=b. \tag{13}$$

At this case, the matrix $\boldsymbol{A}$ is non-singular and positive definite, the matrix $\boldsymbol{B}$ is singular and indefinite. In this example, we give the modified shifted Laplace preconditioner:

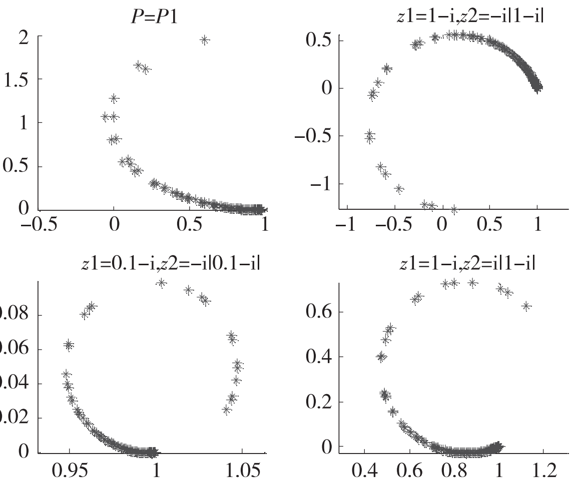

Fig. 1 Spectrum of $\boldsymbol{A}$ and $P^{-1}\boldsymbol{A}$ on 30×30 grid

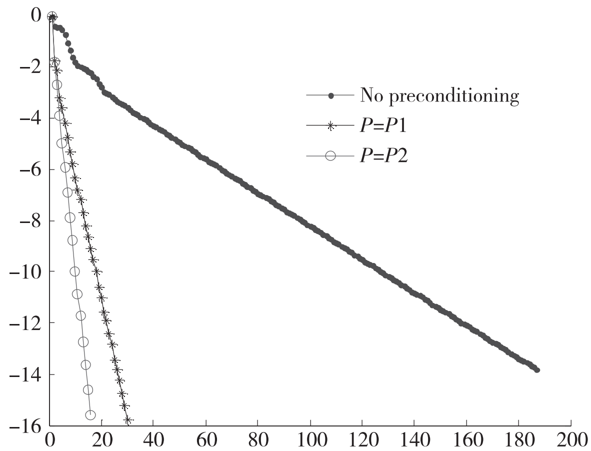

Fig. 2 Convergence curve and total numbers of GMRES iterations on 30×30 grid for example 1

$$P_3=\boldsymbol{A}+\boldsymbol{B}+\mathrm{i}\,|1-\mathrm{i}|\,\boldsymbol{B}.$$

From Fig. 3 and Tab. 2, we can find that the modified shifted Laplace preconditioners can still be applied to the case when $\boldsymbol{B}$ is singular.

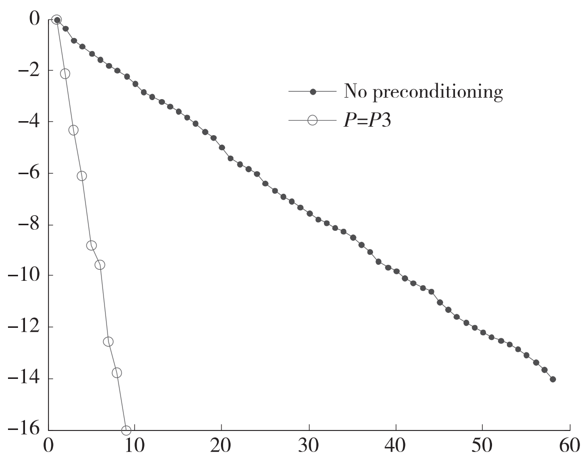

Fig. 3 Convergence curve and total numbers of GMRES iterations on 30×30 grid for example 2

Tab. 2 GMRES iterations for different grids for example 2

Grid	No preconditioning	$P=P3$
10×10	23	5
20×20	43	6
30×30	57	8
40×40	73	8

5 Conclusion

In this paper, we introduce the modified shifted Laplace preconditioners, and we give some properties about it, then, we give numerical example to illustrate the efficient of our method.

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对称复线性系统的修正转移 Laplace 预条件子

何　军，刘衍民，田俊康，张转周

（遵义师范学院 数学与计算科学学院，贵州 遵义 563002）

摘要：介绍了修正转移 Laplace 预条件子来解决对称复线性系统，这类系统常常是不定的、大型的，并且用迭代的方法来求解很困难。研究了预处理子的性质，表明预处理后的矩阵的特征值变得非常集中。数值例子阐述了预条件子的有效性。

关键词：预条件子；复线性系统；子空间迭代

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