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带有 Markov 链利率的相依风险模型破产概率的界*

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摘要:【目的】研究具有齐次马氏链利率过程的离散时间相依风险模型。【方法】针对保费在期初收取和索赔在期末收取的情况,利用鞅方法和更新迭代的方法对破产概率模型进行研究,其中利率的马尔可夫性体现了未来利率与历史利率的独立性,保费过程和索赔过程都具有高阶自回归结构,考虑两者各自的相依性可使模型更具有普遍性。【结果】得到了所研究风险模型破产概率的上界。【结论】研究结果为解决实际生活中的保险问题提供了一定的理论依据。

关键词:Markov 链;高阶自回归结构;离散时间风险模型;破产概率;鞅方法
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在经典风险模型中,一般都假设各个随机变量之间是相互独立的,但这有时候会与实际情况不符。随着经济多元化的发展和保险公司对其保费的再投资等因素的影响,利率呈现了 Markov 性,因而突出了马氏链利率在风险模型中的重要性。由于多种风险的影响,如灾难性风险、市场风险、保险公司营运风险等风险,使得保费过程或索赔过程出现一定的相依性,因而相依风险模型的关注度便越来越高。Yang 等人[1]讨论了常利率下的相依风险情况,其中保费过程和索赔过程都仅限于一阶自回归结构($AR(1)$);Xu 等人[2]研究了利率为 Markov 链,纯损失具有 $AR(1)$结构的离散时间风险模型;彭江艳等人[3]研究了常利率下广义连续时间更新风险模型的最终破产概率,其中索赔过程为一个相依的高阶自回归过程;彭江艳还详细研究了常利率下相依保险风险模型[4];程建华等人[5]研究了利率为齐次 Markov 链,保费过程和索赔过程都具有一阶自回归结构($AR(1)$)的离散时间风险模型,运用鞅方法得到破产概率的上界;He 等人[6]考虑了带随机利率的离散时间风险模型,其中利率为马氏链,保费过程和索赔过程都是独立同分布的随机变量序列;郭风龙等人[7]研究了一类离散时间风险模型的破产概率,其中保费过程和利率过程为离散时间 Markov 链,索赔过程为独立同分布的非负随机变量序列。吕海娟等人[8]研究了随机利率下高阶自回归风险模型的破产概率。

本文将考虑利率为 Markov 链,保费过程和索赔过程都是高阶自回归结构。高阶自回归结构在时间序列分析中比一阶自回归结构更具有普遍性。在实际生活中,索赔额不仅仅与前一个阶段相关,还可能与以往 n 个阶段相关,而多个初始值会带来相应的数学证明困难和技巧的复杂性。文中针对保费在期初收取和索赔在期末收取的情况,运用鞅方法得出破产概率的界。

1 模型的描述

考虑模型 $U_n(x)=u\prod_{i=1}^{n}(1+Z_i)+\sum_{i=1}^{n}X_i\prod_{k=i}^{n}(1+Z_k)-\sum_{i=1}^{n}Y_i\prod_{k=i+1}^{n}(1+Z_k)$。在该模型中有:

1) u 为保险公司的初始准备金,$u\geqslant 0$;U_n 表示保险公司到 n 时刻的盈余。

2) Z_i 表示$(i-1,i]$时段内的利率,是非负的随机变量序列,$\{Z_n,n=1,2,\cdots\}$是一个齐次马氏链,其状态空间为 $E=\{z_s,s=0,1,2,\cdots\}$,转移概率为 $P_{ab}=P\{Z_{n+1}=z_t\mid Z_n=z_s\}$,其中,$a,b$ 为在状态空间任取的。初始值 $Z_0=z_0$,$z=\min\{z_s,s=0,1,2,\cdots\}$,$\tilde{z}=\max\{z_s,s=0,1,2,\cdots\}$。

3) X_i 表示$(i-1,i]$时段内的保费,服从高阶自回归 $AR(p)$ 模型,即 $X_n=a_1X_{n-1}+a_2X_{n-2}+\cdots+a_pX_{n-p}+\delta_n$,

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$n=1,2,\cdots$,其中 $0\leqslant a_i<1,X_{1-i}=x_{1-i}\geqslant 0,i=1,\cdots,p,x_0,x_{-1},\cdots,x_{1-p}$ 为保费以前 p 年的数据,$\{\delta_n,n=1,2,\cdots\}$为独立同分布的非负随机变量序列。

4) Y_i 表示$(i-1,i]$时段内的索赔额,服从高阶自回归$AR(p)$模型,即 $Y_n=b_1Y_{n-1}+b_2Y_{n-2}+\cdots+b_pY_{n-p}+\varepsilon_n$,$n=1,2,\cdots$,其中 $0\leqslant b_i<1,Y_{1-i}=y_{1-i}\geqslant 0,i=1,\cdots,p,y_0,y_{-1},\cdots,y_{1-p}$ 为索赔额以前 p 年的数据,$\{\varepsilon_n,n=1,2,\cdots\}$为独立同分布的非负的随机变量序列。

5) 保费在时间区间初交纳,理赔在时间区间末进行。$\{Z_n,n=1,2,\cdots\}$,$\{X_n,n=1,2,\cdots\}$,$\{Y_n,n=1,2,\cdots\}$相互独立。设 $T=\inf\{n:U_n<0,n>0\}(\inf\{\varnothing\}=\infty)$表示破产时刻,定义破产概率为

$$\varphi(u,x_0,x_{-1},\cdots,x_{1-p},y_0,y_{-1},\cdots,y_{1-q},z_0)=$$
$$P\{T<\infty|\,U_0=u,X_0=x_0,\cdots,X_{1-p}=x_{1-p},Y_0=y_0,\cdots,Y_{1-p}=y_{1-p},Z_0=z_0\}=$$
$$P\left\{\bigcup_{n=1}^{\infty}(U_n<0)\mid U_0=u,X_0=x_0,\cdots,X_{1-p}=x_{1-p},Y_0=y_0,\cdots,Y_{1-p}=y_{1-p},Z_0=z_0\right\}。$$

2 破产概率的界

引理1 如果 $\dfrac{E(\varepsilon)}{(1+z)\left(1-\sum\limits_{i=1}^{p}b_i(1+z)^{-i}\right)}<\dfrac{E(\delta)}{1-\sum\limits_{i=1}^{p}a_i(1+\tilde{z})^{-i}}$ 成立,那么存在唯一的正数 R,使得

$$E\left(\exp\left\{-R\left[\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right]\right\}\right)=1。\tag{1}$$

证明 考虑函数 $f(\theta)=E\left(\exp\left\{\theta\left[\dfrac{\varepsilon_n(1+z)^{-1}}{1-\sum\limits_{i=1}^{p}b_i(1+z)^{-i}}-\dfrac{\delta_n}{1-\sum\limits_{i=1}^{p}a_i(1+\tilde{z})^{-i}}\right]\right\}\right)-1$,求导可知 $f''(\theta)\geqslant 0$,所以 $f(\theta)$ 是凸函数。又因为 $f(0)=0,f'(0)<0$,故 R 存在且是唯一的。 证毕

定理1 如果 $u\geqslant 0,0<z\leqslant\tilde{z}<\infty,a_i\geqslant b_i,\sum\limits_{i=1}^{p}\dfrac{b_i}{(1+z)^i}<1,i=1,2,\cdots p$,且(1)式成立,则有

$$\varphi(u,x_0,x_{-1},\cdots,x_{1-p},y_0,y_{-1},\cdots,y_{1-p},z_0)=P\{T<\infty\}\leqslant\frac{\mathrm{e}^{-RW_0}}{E(\mathrm{e}^{-RW_T}\mid T<\infty)},$$

其中

$$W_0=u+\sum_{i=1}^{p}c_i\prod_{m=1}^{i}(1+Z_m)x_{1-i}-\sum_{j=1}^{p}d_j\prod_{m=1}^{j-1}(1+Z_m)y_{1-j},$$
$$W_n=U_n\prod_{i=1}^{n}(1+Z_i)^{-1}+\sum_{i=1}^{p}c_i\prod_{m=1}^{n-i}(1+Z_m)^{-1}X_{n-i+1}-\sum_{j=1}^{p}d_j\prod_{m=1}^{n-j+1}(1+Z_m)^{-1}Y_{n-j+1},$$
$$c_i=\frac{\sum_{j=i}^{p}a_j\prod_{m=1}^{j}(1+Z_m)^{-1}}{1-\sum_{i=1}^{p}a_i\prod_{m=1}^{i}(1+Z_m)^{-1}},d_j=\frac{\sum_{i=j}^{p}b_i\prod_{m=1}^{i}(1+Z_m)^{-1}}{1-\sum_{i=1}^{p}b_i\prod_{m=1}^{i}(1+Z_m)^{-1}},i,j=1,2,\cdots,p。$$

证明 构造 $W_n=U_n\prod\limits_{i=1}^{n}(1+Z_i)^{-1}+\sum\limits_{i=1}^{p}c_i\prod\limits_{m=1}^{n-i}(1+Z_m)^{-1}X_{n-i+1}-\sum\limits_{j=1}^{p}d_j\prod\limits_{m=1}^{n-j+1}(1+Z_m)^{-1}Y_{n-j+1}$,则可得 W_0:

$$W_0=u+\sum_{i=1}^{p}c_i\prod_{m=1}^{i}(1+Z_m)x_{1-i}-\sum_{j=1}^{p}d_j\prod_{m=1}^{j-1}(1+Z_m)y_{1-j}=$$
$$u+\sum_{i=1}^{p}\frac{\sum_{j=i}^{p}a_j\prod_{m=1}^{j}(1+Z_m)^{-1}}{1-\sum_{j=1}^{p}a_j\prod_{m=1}^{j}(1+Z_m)^{-1}}\prod_{m=1}^{i}(1+Z_m)x_{1-i}-\sum_{j=1}^{p}\frac{\sum_{i=j}^{p}b_i\prod_{m=1}^{i}(1+Z_m)^{-1}}{1-\sum_{i=1}^{p}b_i\prod_{m=1}^{i}(1+Z_m)^{-1}}\prod_{m=1}^{j-1}(1+Z_m)y_{1-j}\geqslant$$
$$u+\sum_{i=1}^{p}\frac{\sum_{j=i}^{p}a_j(1+\tilde{z})^{-(j-i)}}{1-\sum_{j=1}^{p}a_j(1+\tilde{z})^{-j}}x_{1-i}-\sum_{j=1}^{p}\frac{\sum_{i=j}^{p}b_i(1+z)^{-(i-j+1)}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}y_{1-j},$$

对 W_n 运用递推法和放缩法,可得如下过程:

$$W_n = U_{n-1}\prod_{i=1}^{n-1}(1+Z_i)^{-1} + X_n\prod_{i=1}^{n-1}(1+Z_n)^{-1} - Y_n\prod_{i=1}^{n}(1+Z_i)^{-1} + \sum_{i=1}^{p}c_i\prod_{m=1}^{n-i}(1+Z_m)^{-1}X_{n-i+1} -$$

$$\sum_{j=1}^{p}d_j\prod_{m=1}^{n-j+1}(1+Z_m)^{-1}Y_{n-j+1} = U_{n-1}\prod_{i=1}^{n-1}(1+Z_i)^{-1} + (1+c_1)\prod_{m=1}^{n-1}(1+Z_m)^{-1}X_n + \sum_{i=2}^{p}c_i\prod_{m=1}^{n-i}(1+Z_m)^{-1}X_{n-i+1} -$$

$$(1+d_1)\prod_{m=1}^{n}(1+Z_m)^{-1}Y_n - \sum_{j=2}^{p}d_j\prod_{m=1}^{n-j+1}(1+Z_m)^{-1}Y_{n-j+1} =$$

$$U_{n-1}\prod_{i=1}^{n-1}(1+Z_i)^{-1} + \sum_{i=1}^{p}a_i(1+c_1)\prod_{m=1}^{n-1}(1+Z_m)^{-1}X_{n-i} + \sum_{i=1}^{p-1}c_{i+1}\prod_{m=1}^{n-i-1}(1+Z_m)^{-1}X_{n-i} -$$

$$\sum_{j=1}^{p}b_j(1+d_1)\prod_{m=1}^{n}(1+Z_m)^{-1}Y_{n-j} - \sum_{j=1}^{p-1}d_{j+1}\prod_{m=1}^{n-j}(1+Z_m)^{-1}Y_{n-j} + (1+c_1)\prod_{m=1}^{n-1}(1+Z_m)^{-1}\delta_n -$$

$$(1+d_1)\prod_{m=1}^{n}(1+Z_m)^{-1}\varepsilon_n = U_{n-1}\prod_{i=1}^{n-1}(1+Z_i)^{-1} + \sum_{i=1}^{p}c_i\prod_{m=1}^{n-i-1}(1+Z_m)^{-1}X_{n-i} - \sum_{j=1}^{p}d_j\prod_{m=1}^{n-j}(1+Z_m)^{-1}Y_{n-j} +$$

$$(1+c_1)\prod_{m=1}^{n-1}(1+Z_m)^{-1}\delta_n - (1+d_1)\prod_{m=1}^{n}(1+Z_m)^{-1}\varepsilon_n = W_{n-1} + (1+c_1)\prod_{m=1}^{n-1}(1+Z_m)^{-1}\delta_n -$$

$$(1+d_1)\prod_{m=1}^{n}(1+Z_m)^{-1}\varepsilon_n = W_{n-1} + \frac{\delta_n\prod_{m=1}^{n-1}(1+Z_m)^{-1}}{1-\sum_{i=1}^{p}a_i\prod_{m=1}^{i}(1+Z_m)^{-1}} - \frac{\varepsilon_n\prod_{m=1}^{n}(1+Z_m)^{-1}}{1-\sum_{i=1}^{p}b_i\prod_{m=1}^{i}(1+Z_m)^{-1}} \geqslant$$

$$W_{n-1} + \left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}} - \frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)\prod_{m=1}^{n-1}(1+Z_m)^{-1},$$

令 $V_n = \mathrm{e}^{-RW_n}$，构造 σ-域 $F_n = \sigma(Z_1, Z_2, \cdots, Z_n, \delta_1, \delta_2, \cdots, \delta_n, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$。

先证明 V_n 是关于 $F_n = \sigma(Z_1, \cdots Z_n, \delta_1, \cdots, \delta_n, \varepsilon_1, \cdots, \varepsilon_n)$ 的上鞅，

$$E(V_n \mid F_{n-1}) = E(\mathrm{e}^{-RW_n} \mid F_{n-1}) \leqslant E\left(\mathrm{e}^{-R\left[W_{n-1}+\left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)\prod_{m=1}^{n-1}(1+Z_m)^{-1}\right]} \middle| F_{n-1}\right) =$$

$$V_{n-1}E\left(\mathrm{e}^{-R\left[\left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)\prod_{m=1}^{n-1}(1+Z_m)^{-1}\right]} \middle| F_{n-1}\right), \tag{2}$$

由于 $\prod_{m=1}^{n-1}(1+Z_m)^{-1} \in F_{n-1}$，且 $\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}} - \frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}$ 与 F_{n-1} 独立，所以

$$E\left(\mathrm{e}^{-R\left[\left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)\prod_{m=1}^{n-1}(1+Z_m)^{-1}\right]} \middle| F_{n-1}\right) = g\left(\prod_{m=1}^{n-1}(1+Z_m)^{-1}\right),$$

其中

$$g(x) = E\left(\mathrm{e}^{-R\left[\left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)x\right]} \middle| F_{n-1}\right) = E\left(\mathrm{e}^{-R\left[\left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)x\right]}\right)$$

易知 $0 < \prod_{m=1}^{n-1}(1+Z_m)^{-1} \leqslant 1$，根据 Jensen 不等式和(5)式可得 $g(x) \leqslant E\left(\mathrm{e}^{-R\left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)}\right)^x = 1$，

所以得到：

$$g\left(\prod_{m=1}^{n-1}(1+Z_m)^{-1}\right) \leqslant 1, \tag{3}$$

将(3)式代入(2)式可得：

$$V_{n-1}E\left(\mathrm{e}^{-R\left[\left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)\prod_{m=1}^{n-1}(1+Z_m)^{-1}\right]} \middle| F_{n-1}\right) \leqslant V_{n-1}E\left(\mathrm{e}^{-R\left[\left(\frac{\delta_n}{1-\sum_{i=1}^{p}a_i(1+\tilde{z})^{-i}}-\frac{\varepsilon_n(1+z)^{-1}}{1-\sum_{i=1}^{p}b_i(1+z)^{-i}}\right)\right]}\right)\cdot$$

$$\prod_{m=1}^{n-1}(1+Z_m)^{-1}=V_{n-1}。$$

则 V_n 是关于 F_n 的上鞅。因为破产时刻 T 是停时，容易得到 $T\wedge n$ 是有界停时，由有界停时定理可得 $E(\mathrm{e}^{-RW_{T\wedge n}})\leqslant E(\mathrm{e}^{-RW_0})=\mathrm{e}^{-RW_0}$，进一步可证

$$\mathrm{e}^{-RW_0}\geqslant E(\mathrm{e}^{-RW_{T\wedge n}})=E(\mathrm{e}^{-RW_{T\wedge n}}I_{\{T\leqslant n\}})+E(\mathrm{e}^{-RW_{T\wedge n}}I_{\{T>n\}})\geqslant E(\mathrm{e}^{-RW_{T\wedge n}}I_{\{T\leqslant n\}})=E(\mathrm{e}^{-RW_T}\mid T\leqslant n)P(T\leqslant n)。$$

令 $n\to\infty$ 取极限得 $\mathrm{e}^{-RW_0}\geqslant E(\mathrm{e}^{-RW_T}\mid T<\infty)P(T<\infty)$，进一步可得 $P(T<\infty)\leqslant\dfrac{\mathrm{e}^{-RW_0}}{E(\mathrm{e}^{-RW_T}\mid T<\infty)}$。 证毕

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Bounds for Ruin Probability in a Dependent Risk Model with a Markov Chain Interest Rate

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Abstract: [**Purposes**]It investigates a discrete time dependent risk model where the interest rate process is homogeneous Markov chain. [**Methods**]The premiums are received at the beginning of each period and the claims are obtained at the end of each period in the model, the model was studied by using martingale method and renewal the iterative method. The Markov chain interest reflects the independence of the further rates and historic rates. The premiums process and the claims process have higher-order autoregressive structure. It can make the model more universal to consider their dependence. [**Findings**]The upper bound of infinite time ruin probabilities derived by using martingale method and renewal the iterative method. [**Conclusions**]The results provide certain theoretical base for the insurance problem in real life.

Keywords: Markov chain; higher-order autoregressive structure; discrete time risk model; ruin probability; martingale method

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