2017 年度重庆市出版专项基金资助栏目　　DOI:10.11721/cqnuj20180302

Operations Research and Cybernetics

Nonconvex Separation Theorems via Assumption B and Applications in Vector Optimization*

ZHAO Kequan, ZHU Qiao, WANG Dingguo

(School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China)

Abstract: [**Purposes**]Nonconvex separation theorems have been playing a very important role in studying nonlinear scalarization technique for vector optimization problems. [**Methods**]By means of the Gerstewitz nonlinear scalarization function and Assumption B, nonconvex separation theorems and applications in vector optimization are studied. [**Findings**]Some nonconvex separation theorems are developed via Assumption B with generalized interiors and some special cases are discussed such as for the free disposal sets and the co-radiant sets. As applications, some nonlinear scalarization results are obtained for several kinds of weakly efficient solutions with generalized interiors in vector optimization. Moreover, some examples are also presented to illustrate the main results. [**Conclusions**]A new method is provided for studying the optimization problems with the ordering cone having empty interior or empty relative interior or even empty relative algebraic interior.

Keywords: nonconvex separation theorems; generalized interiors; weakly efficient solutions; nonlinear scalarization; vector optimization

CLC number: O221.6　　**Document code**: A　　**Article number**: 1672-6693(2018)03-0001-08

It is well known that the theory and methods of vector optimization have been playing an important role in many fields. In vector optimization, one of the most important research topics is how to characterize various kinds of exact and approximate solutions. So far there have been a lot of fundamental results following the different techniques[1-3].

Linear scalarization technique has been widely applied to study vector optimization problems[4-6]. Aiming at those vector optimization problems with the ordering cones having possibly empty topological interior, and by means of the relative algebraic interior proposed by Holmes[7], Adán and Novo[8] introduced new notions of weak efficient solutions and obtained some linear scalarization characterizations. Furthermore, Limber and Goodrich[9] introduced the quasi interior in a normed linear space. Borwein and Lewis[10] introduced the quasi relative interior, which is an extension of both the relative topological interior in some finite dimensional spaces and the quasi interior. More properties of the quasi interior and quasi relative interior can be found in [11-13].

Nonlinear scalarization methods also have been paid more attentions in vector optimization. By Gerstewitz

* **Received**: 06-15-2017　**Accepted**: 01-12-2018　**Internet publishing time**: 05-22-2018　10:02

Fundation: Supported by the Key Program of National Natural Science Foundation of China(No. 11431004); the General Program of National Natural Science Foundation of China(No. 11671062; No. 11271391); Chongqing Basic and Frontier Research Project(No. cstc2015jcyjA00027; No. cstc2014jcyjA00004) and Science and Technology Research Project of the Chongqing Municipal Education Committee(No. KJ1500303)

The first author biography: ZHAO Kequan, male, professer, Ph.D. supervisor, research area is the theory and method of vector optimization, E-mail: kequanz@163.com

收稿日期:2017-06-15　**修回日期**:2018-01-12　**网络出版时间**:2018-05-22　10:02

资助项目:国家自然科学基金重点项目(No. 11431004);国家自然科学基金面上项目(No. 11671062;No. 11271391);重庆市基础与前沿研究计划项目(No. cstc2015jcyjA00027;No. cstc2014jcyjA00004);重庆市教委科学技术研究项目(No. KJ1500303)

第一作者简介:赵克全,男,教授,博士生导师,研究方向为向量优化理论与方法,E-mail:kequanz@163.com

网络出版地址:http://kns.cnki.net/kcms/detail/50.1165.N.20180522.1002.044.html

nonlinear scalarization functional, Göpfert et al.[14] developed the classical nonconvex separation theorems and obtained some nonlinear scalarization results of approximate (weak) efficient solutions of vector optimization problems. Furthermore, Gutiérrez et al.[15] obtained some nonlinear scalarization results of (C,ε)-efficient solutions. Gao et al.[16] proposed a kind of proper efficient solutions via co-radiant sets and obtained some nonlinear scalarization results. Chicco et al.[17] introduced the concept of improvement sets which is close to the free disposal sets, and gave some properties of E-efficient solutions defined by improvement sets. Zhao et al.[18] established some nonlinear scalarization results of E-efficient solutions. More properties of improvement sets can be found in [19-21]. Improvement sets and co-radiant sets are very important tools to study vector optimization problems in a unified framework. Flores-Bazán and Hernández[22] presented Assumption B and obtained some complete nonlinear scalarization results for vector optimization problems.

Based on the fact that the classical nonconvex separation theorems given so far in the literatures rely heavily on the nonemptiness of the topological interior and cannot be applied for those optimization problems with the sets having possibly empty interior, empty relative interior and even empty relative algebraic interior. Motivated by the works of [7,10,14,22], in this paper, we mainly establish some nonconvex separation theorems via Assumption B with generalized interiors and discuss some special cases for the free disposal sets and the co-radiant sets. Furthermore, we also obtain some nonlinear scalarization results of several kinds of approximate solutions with generalized interiors for vector optimization problems.

1 Preliminaries

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space, $\mathbf{R}_+$ be the set of all nonegative real numbers, $\mathbf{R}_{++}$ be the set of all positive real numbers, $\mathbf{R}_{++}q=\{\alpha q\mid\alpha\in\mathbf{R}_{++}\}$, $\mathbf{N}^+$ be the set of all positive integers.

Let Y be a real linear space. For a nonempty subset A in Y, the cone hull of A is defined as $\mathrm{cone}A=\{\alpha a\mid \alpha\in\mathbf{R}_+,a\in A\}$. The relative algebraic interior[7] and the vector closure[8] of A denoted by $\mathrm{icr}A$ and $\mathrm{vcl}A$ are respectively defined as

$$\mathrm{icr}A=\{y\in A\mid \forall\, y'\in\mathrm{aff}A-y,\ \exists\,\delta'>0,\ \forall\,\delta\in[0,\delta'],y+\delta y'\in A\},$$

$$\mathrm{vcl}A=\{y\in Y\mid \exists\, y'\in Y,\ \forall\,\delta'>0,\ \exists\,\delta\in[0,\delta'],y+\delta y'\in A\}.$$

Let Y be a real topological linear space. The relative interior of A denoted by $\mathrm{ri}A$ is defined as

$$\mathrm{ri}A=\{y\in A\mid \exists\text{ a neighborhood of zero }U\text{ such that }(y+U)\cap\mathrm{cl}(\mathrm{aff}A)\subset A\}.$$

For a nonempty convex subset A in Y, the quasi interior[9] and quasi relative interior[10] denoted by $\mathrm{qi}A$ and $\mathrm{qri}A$ are respectively defined as

$$\mathrm{qi}A=\{y\in A\mid\mathrm{cl}(\mathrm{cone}(A-y))=Y\},\mathrm{qri}A=\{y\in A\mid\mathrm{cl}(\mathrm{cone}(A-y))\text{ is a linear subspace of }Y\}.$$

Assume that K be a nonempty convex cone in Y. We say that K is pointed if $K\cap(-K)=\{0\}$. We say that a nonempty subset A in Y is a free disposal set with respect to K if $A+K=A$. A is said to be an improvement set with respect to K if A is a free disposal set with respect to K and $0\notin A$[17,19]. We say that A is a co-radiant set[15] if A satisfies $\alpha d\in A$ for every $d\in A$, $\alpha>1$, and we denote

$$C(\varepsilon)=\varepsilon C,\ \forall\,\varepsilon>0,C(0)=\bigcup_{\varepsilon>0}C(\varepsilon).$$

We say that A is star-shaped if there exists $q\in A$ such that for all $y\in A$ and $\alpha\in(0,1),\alpha q+(1-\alpha)y\in A$. We denote by $\mathrm{kern}A$ the set of all points $q\in A$. Moreover, we say that A is a proper set if $A\neq\varnothing$ and $A\neq Y$.

In this paper, we shall use the following Gerstewitz nonlinear scalarization functional $\varphi_{q,S}:Y\to\mathbf{R}\cup\{\pm\infty\}$: $\varphi_{q,S}(y)=\inf\{t\in\mathbf{R}\mid y\in tq-S\}$, where S is a proper set in Y, $y\in Y$ and $0\neq q\in Y$. We denote $\inf\varnothing=+\infty$.

In order to study vector optimization problems in a unified framework, Flores-Bazán and Hernández[22] proposed a general assumption condition named as Assumption B as follows:

Assume that Y be a real topological linear space, $0\neq q\in Y,S\subset Y$ be a proper set with nonempty topological interior and $\mathrm{cl}S+\mathbf{R}_{++}q\subset\mathrm{int}S$.

Remark 1 (i) Let S be closed and there exists a cone K such that $q\in \mathrm{int}K$ and $S+\mathrm{int}K\subset S$. Göpfert et al.[14] proved that S satisfies Assumption B.

(ii) Let S be a nonempty improvement set with respect to a convex cone K and $0\neq q\in \mathrm{int}K$. Zhao and Yang[20] proved that S satisfies Assumption B.

(iii) Let S be a proper star-shaped co-radiant set and $0\neq q\in \mathrm{int}(\mathrm{kern}S)$. Gutiérrez et al.[15] proved that S satisfies Assumption B.

In this paper, we will use the following Assumption B via generalized interiors.

(i) Assumption B via relative algebraic interior: Y is a real linear space, $0\neq q\in Y$, $S\subset Y$ is a proper set with $\mathrm{icr}S\neq\emptyset$ and $\mathrm{vcl}S+\mathbf{R}_{++}q\subset \mathrm{icr}S$.

(ii) Assumption B via relative interior: Y is a real topological linear space, $0\neq q\in Y$, $S\subset Y$ is a proper set with $\mathrm{ri}S\neq\emptyset$ and $\mathrm{cl}S+\mathbf{R}_{++}q\subset \mathrm{ri}S$.

(iii) Assumption B via quasi interior: Y is a real separated locally convex topological linear space, $0\neq q\in Y$, $S\subset Y$ is a proper convex set with $\mathrm{qi}S\neq\emptyset$ and $\mathrm{cl}S+\mathbf{R}_{++}q\subset \mathrm{qi}S$.

(iv) Assumption B via quasi relative interior: Y is a real separated locally convex topological linear space, $0\neq q\in Y$, $S\subset Y$ is a proper convex set with $\mathrm{qri}S\neq\emptyset$ and $\mathrm{cl}S+\mathbf{R}_{++}q\subset \mathrm{qri}S$.

2 Separation theorems via relative algebraic interior

By means of Assumption B, Flores-Bazán and Hernández[22] obtained the following nonconvex separation theorem.

Theorem 1 If S satisfies Assumption B, then for all $\lambda\in\mathbf{R}$, (i) $\{y\in Y\mid \varphi_{q,S}(y)<\lambda\}=\lambda q-\mathrm{int}S$; (ii) $\{y\in Y\mid \varphi_{q,S}(y)\leqslant\lambda\}=\lambda q-\mathrm{cl}S$.

Remark 2 In Theorem 1, the set S may not be closed and also may not be a cone. If S is a closed convex cone and $q\in \mathrm{int}S$, then Theorem 1 coincides with Proposition 1.43 in [2].

Remark 3 If $\mathrm{cl}S+\mathbf{R}_{++}q\subset \mathrm{int}S$ is replaced by $\mathrm{cl}S+\mathbf{R}_{++}q\subset S$, then Theorem 1 does not necessarily be true.

Example 1 Let $Y=\mathbf{R}^2$, $q=(0,1)$, $\lambda=0$ and

$$S=\{(y_1,y_2)\mid y_1+y_2\geqslant 1, y_1\geqslant 0, y_2\geqslant 0\}\cup\{(y_1,y_2)\mid y_1=0, 0\leqslant y_2\leqslant 1\}.$$

We observe that $\mathrm{cl}S=S$, $\mathrm{cl}S+\mathbf{R}_{++}q\not\subset \mathrm{int}S$ and $\mathrm{cl}S+\mathbf{R}_{++}q\subset S$. Clearly,

$$\{y=(y_1,y_2)\in Y\mid \varphi_{q,S}(y)<0\}=-\mathrm{cl}S-\mathbf{R}_{++}q\neq -\mathrm{int}S.$$

The above simple example indicates that the assumption condition $\mathrm{cl}S+\mathbf{R}_{++}q\subset S$ could be too nonrestrictive to ensure validity of the nonconvex separation theorem and hence this motivates us to consider Assumption B via generalized interiors. In the following, we first focus on nonconvex separation theorems by making use of Assumption B via relative algebraic interior.

Theorem 2 If S satisfies Assumption B via relative algebraic interior, then for all $\lambda\in\mathbf{R}$, (i) $\{y\in Y\mid \varphi_{q,S}(y)<\lambda\}=\lambda q-\mathrm{icr}S$; (ii) $\{y\in Y\mid \varphi_{q,S}(y)\leqslant\lambda\}=\lambda q-\mathrm{vcl}S$.

Proof We first prove (i). Let $y_0\in\lambda q-\mathrm{icr}S$. Then for any $\varepsilon>0$,

$$\lambda q-y_0+\varepsilon q\in \mathrm{vcl}S+\mathbf{R}_{++}q\subset \mathrm{icr}S\subset S.$$

Thus $-\varepsilon q\in \mathrm{aff}S-(\lambda q-y_0)$. By $\lambda q-y_0\in \mathrm{icr}S$, there exists $\delta'>0$ such that $\lambda q-y_0-\frac{\delta'\varepsilon}{2}q\in S$ implies $\varphi_{q,S}(y_0)=\inf\{t\in\mathbf{R}\mid y_0\in tq-S\}\leqslant\lambda-\frac{\delta'\varepsilon}{2}<\lambda$.

On the other hand, assume that $y_0\in\{y\in Y\mid \varphi_{q,S}(y)<\lambda\}$. Then there exists $t_0\in\mathbf{R}$ such that $t_0<\lambda$ and $y_0\in t_0q-S$. Consequently, we have

$$y_0\in t_0q-S=\lambda q-((\lambda-t_0)q+S)\subset\lambda q-(\mathrm{vcl}S+\mathbf{R}_{++}q)\subset\lambda q-\mathrm{icr}S.$$

Next, we prove (ii). Assume that $y_0\in\{y\in Y\mid \varphi_{q,S}(y)\leqslant\lambda\}$.

If $\varphi_{q,S}(y_0)<\lambda$, then by (i) we have $y_0\in\lambda q-\mathrm{icr}S\subset\lambda q-\mathrm{vcl}S$.

If $\varphi_{q,S}(y_0)=\lambda$, then $y_0\in\lambda q-\mathrm{vcl}S$. In fact, from $\varphi_{q,S}(y_0)=\lambda$, it follows that for any $\frac{1}{n}>0$, $n\in\mathbf{N}^+$, there exist $t_n\in\mathbf{R}$ such that $\lambda\leqslant t_n<\lambda+\frac{1}{n}$ and

$$y_0\in t_nq-S. \tag{1}$$

Case 1: Assume that there exists $n_0\in\mathbf{N}^+$ such that $t_{n_0}=\lambda$, then $y_0\in t_{n_0}q-S=\lambda q-S\subset\lambda q-\mathrm{vcl}S$.

Case 2: Assume that for any $n\in\mathbf{N}^+$, $\lambda<t_n<\lambda+\frac{1}{n}$. It is clear that $t_n\to\lambda(n\to\infty)$. Since $\lambda<t_n$, there exists $\delta_n>0$ such that $t_n=\lambda+\delta_n$ and so $\delta_n=t_n-\lambda(n\to\infty)$. By combining this with (1), we get $y_0\in(\lambda+\delta_n)q-S$ and $\lambda q-y_0+\delta_nq\in S, n=1,2,\cdots$. For any $\delta'>0$, there exists $N\in\mathbf{N}^+$ such that for any $n>N$, $0<\delta_n<\delta'$ and $\lambda q-y_0+\delta_nq\in S$. It follows that $y_0\in\lambda q-\mathrm{vcl}S$.

On the other hand, suppose that $y_0\in\lambda q-\mathrm{vcl}S$, then we have for any $\varepsilon>0$, $\lambda q-y_0+\varepsilon q\in\mathrm{vcl}S+\mathbf{R}_{++}q\subset\mathrm{icr}S\subset S$, i.e., $y_0\in(\lambda+\varepsilon)q-S$. Hence from the arbitrariness of ε, we get $\varphi_{q,S}(y_0)\leqslant\lambda$.

Corollary 1 Let Y be a real linear space, K be a convex cone in Y and $0\neq q\in\mathrm{icr}K$. If E is a vector closed free disposal set with respect to K and $E\subset K$, then for all $\lambda\in\mathbf{R}$, (i) $\{y\in Y|\varphi_{q,E}(y)<\lambda\}=\lambda q-\mathrm{icr}E$; (ii) $\{y\in Y|\varphi_{q,E}(y)\leqslant\lambda\}=\lambda q-\mathrm{vcl}E$.

Proof It is clear that for all $\alpha>0$, $\alpha\mathrm{icr}K\subset\mathrm{icr}K$. Hence by Theorem 2, we only need to prove $E+\mathrm{icr}K\subset\mathrm{icr}E$. Let $x\in E+\mathrm{icr}K$. Then there exist $e\in E$ and $k\in\mathrm{icr}K$ such that $x=e+k$, and so $x\in E+K=E\subset K$. Let $h\in\mathrm{aff}E-x$. According to $E\subset K$, one has that $h\in\mathrm{aff}K-x$. Moreover, we can easily show that

$$\mathrm{aff}K-x=\mathrm{aff}K-k. \tag{2}$$

Hence one has further that $h\in\mathrm{aff}K-k$. Furthermore, for $k\in\mathrm{icr}K$, there exists $\bar{\delta}>0$ such that for any $\delta\in[0,\bar{\delta}]$, $k+\delta h\in K$. It follows that $x+\delta h=e+k+\delta h\in e+K\subset E$, which implies $x\in\mathrm{icr}E$. So $E+\mathrm{icr}K\subset\mathrm{icr}E$.

Remark 4 Corollary 1 does not necessarily be valid if $E\subset K$ is removed.

Example 2 Let $Y=\mathbf{R}^2$, $K=\{(y_1,y_2)|y_1=0,y_2\geqslant0\}$ and $E=\{(y_1,y_2)|y_1+y_2\geqslant1,y_1\geqslant0,y_2\geqslant0\}$.

It is clear that E is a proper vector closed set. Furthermore, we can verify that E is a free disposal set with respect to K and $E\not\subset K$. Moreover,

$$\mathrm{icr}K=\{(y_1,y_2)|y_1=0,y_2>0\},\mathrm{icr}E=\{(y_1,y_2)|y_1+y_2>1,y_1>0,y_2>0\}.$$

Let $\lambda=0$ and $q=(0,1)\in\mathrm{icr}K$. Then we have

$$\{y=(y_1,y_2)\in Y|\varphi_{q,E}(y)<0\}=-\mathrm{icr}E\cup\{(y_1,y_2)|y_1=0,y_2<-1\}\neq-\mathrm{icr}E,$$

which implies that Corollary 1 is not true.

Corollary 2 Let Y be a real linear space. If $C\subset Y$ is a vector closed star-shaped co-radiant set, $C\subset\mathrm{cone}(\mathrm{kern}C)$ and $0\neq q\in\mathrm{icr}(\mathrm{cone}(\mathrm{kern}C))$, then for all $\varepsilon>0$ and $\lambda\in\mathbf{R}$, (i) $\{y\in Y|\varphi_{q,C(\varepsilon)}(y)<\lambda\}=\lambda q-\mathrm{icr}C(\varepsilon)$; (ii) $\{y\in Y|\varphi_{q,C(\varepsilon)}(y)\leqslant\lambda\}=\lambda q-\mathrm{vcl}C(\varepsilon)$.

Proof Clearly, $\mathrm{kern}C$ is convex. Then by Corollary 1, we only need to prove

$$C(\varepsilon)+\mathrm{cone}(\mathrm{kern}C)=C(\varepsilon). \tag{3}$$

In fact, this can be obtained immediately by Lemma 5.2(ii) in [15].

3 Separation theorems via relative interior

In this section, we mainly focus on nonconvex separation theorems by making use of Assumption B via relative interior.

Theorem 3 If S satisfies Assumption B via relative interior, then for all $\lambda\in\mathbf{R}$, (i) $\{y\in Y|\varphi_{q,S}(y)<\lambda\}=\lambda q-\mathrm{ri}S$; (ii) $\{y\in Y|\varphi_{q,S}(y)\leqslant\lambda\}=\lambda q-\mathrm{cl}S$.

Proof We first prove (i) Since $\mathrm{cl}S+\mathbf{R}_{++}q\subset\mathrm{ri}S$ implies $\mathrm{vcl}S+\mathbf{R}_{++}q\subset\mathrm{icr}S$, then it follows from the fact that $\mathrm{ri}S\subset\mathrm{icr}S$ and Theorem 2 (i) that

$$\lambda q-\mathrm{ri}S\subset\lambda q-\mathrm{icr}S=\{y\in Y\mid\varphi_{q,S}(y)<\lambda\}.$$

Conversely, let $y_0\in\{y\in Y\mid\varphi_{q,S}(y)<\lambda\}$. Then there exists $t_0<\lambda$ such that

$$y_0\in\lambda q-(S+(\lambda-t_0)q)\subset\lambda q-(S+\mathbf{R}_{++}q)\subset\lambda q-(\mathrm{cl}S+\mathbf{R}_{++}q)\subset\lambda q-\mathrm{ri}S.$$

(ii) It follows immediately by virtue of Corollary 3.3(a) in [22].

Corollary 3 Let Y be a real topological linear space, K be a convex cone in Y and $0\neq q\in\mathrm{ri}K$. If E is a closed free disposal set with respect to K and $E\subset K$, then for all $\lambda\in\mathbf{R}$, (i) $\{y\in Y\mid\varphi_{q,E}(y)<\lambda\}=\lambda q-\mathrm{ri}E$; (ii) $\{y\in Y\mid\varphi_{q,E}(y)\leqslant\lambda\}=\lambda q-\mathrm{cl}E$.

Proof It is clear that $\alpha\mathrm{ri}K\subset\mathrm{ri}K$ for any $\alpha>0$. Then by Theorem 3, we only need to prove $E+\mathrm{ri}K\subset\mathrm{ri}E$. Let $x\in E+\mathrm{ri}K$. Then we have $x\in E+K=E$ and there exist $e\in E$ and $k\in\mathrm{ri}K$ such that $x=e+k$. Since $k\in\mathrm{ri}K$, one has that there exists a neighborhood of zero U such that $(k+U)\cap\mathrm{cl}(\mathrm{aff}K)\subset K$. Then by (2), we have $\mathrm{cl}(\mathrm{aff}K)=\mathrm{cl}(\mathrm{aff}K-e)=\mathrm{cl}(\mathrm{aff}K)-e$. It follows that $(e+k+U)\cap\mathrm{cl}(\mathrm{aff}K)\subset e+K$. Thus, we have

$$(x+U)\cap\mathrm{cl}(\mathrm{aff}E)=(e+k+U)\cap\mathrm{cl}(\mathrm{aff}E)\subset(e+k+U)\cap\mathrm{cl}(\mathrm{aff}K)\subset e+K\subset E.$$

This indicates that $x\in\mathrm{ri}E$.

Corollary 4 Let Y be a real topological linear space. If $C\subset Y$ is a closed star-shaped co-radiant set, $C\subset\mathrm{cone}(\mathrm{kern}C)$ and $0\neq q\in\mathrm{ri}(\mathrm{cone}(\mathrm{kern}C))$, then for all $\varepsilon>0$ and $\lambda\in\mathbf{R}$, (i) $\{y\in Y\mid\varphi_{q,C(\varepsilon)}(y)<\lambda\}=\lambda q-\mathrm{ri}C(\varepsilon)$; (ii) $\{y\in Y\mid\varphi_{q,C(\varepsilon)}(y)\leqslant\lambda\}=\lambda q-\mathrm{cl}C(\varepsilon)$.

Proof It follows immediately by means of the preceding (3) and Corollary 3.

4 Separation theorems via quasi relative interior

In this section, we mainly focus on nonconvex separation theorems by making use of Assumption B via quasi relative interior and quasi interior, respectively.

Theorem 4 If the set S satisfies Assumption B via quasi relative interior, then for all $\lambda\in\mathbf{R}$, (i) $\{y\in Y\mid\varphi_{q,S}(y)<\lambda\}\subset\lambda q-\mathrm{qri}S$; (ii) $\{y\in Y\mid\varphi_{q,S}(y)\leqslant\lambda\}=\lambda q-\mathrm{cl}S$.

Proof The conclusion (i) can be easily obtained similar with the proof of Theorem 3, and (ii) can be derived from $\mathrm{cl}S+\mathbf{R}_{++}q\subset\mathrm{qri}S\subset S$ and Corollary 3.3(a) in [22].

Remark 5 We observe that "$\subset$" may be strict in Theorem 4 (i).

Example 3 Let $Y=l^p$ $(2\leqslant p<+\infty)$ with its usual norm, $l_+^p=\{y=(y_i)_{i\in\mathbf{N}^+}\in l^p\mid y_i\geqslant0, i\in\mathbf{N}^+\}$ and $S=(1,0,0,\cdots)+l_+^p$. We take $\lambda=0$, $q=(1,0,0,\cdots)+\left(\frac{1}{4},\frac{1}{4^2},\frac{1}{4^3},\cdots\right)$, $\bar{y}=(1,0,0,\cdots)+\left(\frac{1}{5},\frac{1}{5^2},\frac{1}{5^3},\cdots\right)$. It is clear that $q\in\mathrm{qri}S$ and $-\bar{y}\in-\mathrm{qri}S$. Moreover, we can prove $-\bar{y}\notin\{y\in Y\mid\varphi_{q,S}(y)<0\}$. Indeed, we only need to prove $\varphi_{q,S}(-\bar{y})\geqslant0$. Equivalently, we need to prove that for all $t<0$, $-\bar{y}\notin tq-S$. On the contrary, assume that there exists $t_0<0$ such that $-\bar{y}\in t_0q-S$. Then it follows that for all $n\in\mathbf{N}^+$, $-\frac{1}{5^n}\leqslant\frac{1}{4^n}t_0$, which means $\left(\frac{4}{5}\right)^n\geqslant-t_0>0$, $\forall n\in\mathbf{N}^+$. This is a contradiction when n is sufficiently large.

Corollary 5 If S satisfies Assumption B via quasi interior, then for all $\lambda\in\mathbf{R}$, (i) $\{y\in Y\mid\varphi_{q,S}(y)<\lambda\}\subset\lambda q-\mathrm{qi}S$; (ii) $\{y\in Y\mid\varphi_{q,S}(y)\leqslant\lambda\}=\lambda q-\mathrm{cl}S$.

Proof The conclusion is trivial because $\mathrm{qri}S=\mathrm{qi}S$ when $\mathrm{qi}S\neq\varnothing$.

Corollary 6 Let Y be a real separated locally convex topological linear space, K be a convex cone in Y and $0\neq q\in\mathrm{qi}K$. If $E\subset Y$ is a closed convex free disposal set with respect to K, then for all $\lambda\in\mathbf{R}$, (i) $\{y\in Y\mid\varphi_{q,E}(y)<\lambda\}\subset\lambda q-\mathrm{qi}E$; (ii) $\{y\in Y\mid\varphi_{q,E}(y)\leqslant\lambda\}=\lambda q-\mathrm{cl}E$.

Proof By Corollary 5, we only need to prove $\mathrm{cl}E+\mathrm{qi}K\subset\mathrm{qi}E$. In fact, this can be easily obtained from $E+K=E$ and Proposition 1 in [11].

Remark 6 If E is not closed, then $\mathrm{cl}E+\mathrm{qi}K\subset\mathrm{qi}E$ may not be valid.

Example 4 Let $Y=l^2$ with its usual norm, $\bar{y}=(n^{-1})_{n\in\mathbf{N}^+}\in Y$, $K=l_+^1=\{y=(y_n)_{n\in\mathbf{N}^+}\in l^1\mid y_n\geqslant0, n\in$

$\mathbf{N}^+\}\subset Y$ and $E=\overline{y}+K$. It is clear that K is a convex cone, E is a convex free disposal set with respect to K and $\mathrm{cl}E=\overline{y}+\mathrm{cl}K=\overline{y}+l^2_+$. From $\overline{y}\in l^2_+$ and $\overline{y}\notin l^1$, we can obtain that $2\overline{y}\in\mathrm{cl}E$ and $2\overline{y}\notin E$, which means that E is not closed. By Example 2 in [11], we have $\mathrm{qri}K=\{y=(y_n)_{n\in\mathbf{N}^+}\in l^1\mid y_n>0,\forall n\in\mathbf{N}^+\}$; $\mathrm{qri}E=\mathrm{qri}(\overline{y}+K)=\overline{y}+\mathrm{qri}K$. Since $2\overline{y}\notin E$, then we have $2\overline{y}\notin\mathrm{qri}E$. Moreover, we can verify $2\overline{y}\in\mathrm{cl}E+\mathrm{qri}K$. In fact, let $\overline{y}'=(n^{-2})_{n\in\mathbf{N}^+}$. Then we have $\overline{y}'\in\mathrm{qri}K$ and $\overline{y}-\overline{y}'\in l^2_+$. It follows that

$$2\overline{y}=\overline{y}+(\overline{y}-\overline{y}')+\overline{y}\in\overline{y}+l^2_++\mathrm{qri}K=\mathrm{cl}E+\mathrm{qri}K.$$

Therefore, $\mathrm{cl}E+\mathrm{qri}K\not\subset\mathrm{qri}E$.

Corollary 7 Let Y be a real separated locally convex topological linear space. If $C\subset Y$ is a proper closed convex co-radiant set and $0\neq q\in\mathrm{qi}(\mathrm{cone}C)$, then for all $\varepsilon>0$ and $\lambda\in\mathbf{R}$, (i) $\{y\in Y\mid\varphi_{q,C(\varepsilon)}(y)<\lambda\}\subset\lambda q-\mathrm{qi}C(\varepsilon)$; (ii) $\{y\in Y\mid\varphi_{q,C(\varepsilon)}(y)\leqslant\lambda\}=\lambda q-\mathrm{cl}C(\varepsilon)$.

Proof It is obvious by (3) and Corollary 6.

5 Applications in vector optimization

Consider the following vector optimization problem:

$$(\mathrm{VOP})\quad\min f(x),\ \mathrm{s.t.}\ x\in D,$$

where $f:X\to Y$ and $D\subset X$. We first introduce several notions of weak efficient solutions of (VOP).

Definition 1 Let S satisfy Assumption B via relative algebraic interior. $\overline{x}\in D$ is said to be a weakly efficient solution of (VOP) if $f(D)\cap(f(\overline{x})-\mathrm{icr}S)=\varnothing$. We denote this by $\overline{x}\in\mathrm{WAE}(f,D,\mathrm{icr}S)$.

Definition 2 Let S satisfy Assumption B via relative topological interior. $\overline{x}\in D$ is said to be a weakly efficient solution of (VOP) if $f(D)\cap(f(\overline{x})-\mathrm{ri}S)=\varnothing$. We denote this by $\overline{x}\in\mathrm{WAE}(f,D,\mathrm{ri}S)$.

Definition 3 Let S satisfy Assumption B via quasi relative interior. $\overline{x}\in D$ is said to be a weakly efficient solution of (VOP) if $f(D)\cap(f(\overline{x})-\mathrm{qri}S)=\varnothing$. We denote this by $\overline{x}\in\mathrm{WAE}(f,D,\mathrm{qri}S)$.

Definition 4 Let S satisfy Assumption B via quasi interior. $\overline{x}\in D$ is said to be a weakly efficient solution of (VOP) if $f(D)\cap(f(\overline{x})-\mathrm{qi}S)=\varnothing$. We denote this by $\overline{x}\in\mathrm{WAE}(f,D,\mathrm{qi}S)$.

Consider the following scalarized optimization problem: $(\mathrm{SOP}_{q,y})\min\limits_{x\in D}\varphi_{q,S}(f(x)-y)$, where $y\in Y$, $0\neq q\in Y$ and $S\subset Y$ is a proper set. We denote $\varphi_{q,S}(f(x)-y)$ by $(\varphi_{q,S,y}\circ f)(x)$. Let $\overline{x}\in D$ and $\varepsilon\geqslant0$. If

$$(\varphi_{q,S,f(\overline{x})}\circ f)(x)\geqslant(\varphi_{q,S,f(\overline{x})}\circ f)(\overline{x})-\varepsilon,\ \forall x\in D,$$

then $\overline{x}$ is called an ε-minimal solution of $(\mathrm{SOP}_{q,f(\overline{x})})$. The set of ε-minimal solutions of $(\mathrm{SOP}_{q,f(\overline{x})})$ is denoted by $\mathrm{Amin}(\varphi_{q,S,f(\overline{x})}\circ f,\varepsilon)$.

Theorem 5 Let S satisfy Assumption B via relative algebraic interior, $0\neq q\in S$, $0\notin\mathrm{icr}S$ and $\varepsilon=\inf\{t\in\mathbf{R}_+\mid tq\in S\}$. Then $\overline{x}\in\mathrm{WAE}(f,D,\mathrm{icr}S)\Leftrightarrow\overline{x}\in\mathrm{Amin}(\varphi_{q,S,f(\overline{x})}\circ f,\varepsilon)$.

Proof Let $\overline{x}\in\mathrm{WAE}(f,D,\mathrm{icr}S)$. It follows that $(f(D)-f(\overline{x}))\cap(-\mathrm{icr}S)=\varnothing$. By Theorem2 (i), we have

$$\{y\in Y\mid\varphi_{q,S}(y)<0\}=-\mathrm{icr}S.\tag{4}$$

Combine with (4), we have $(f(D)-f(\overline{x}))\cap\{y\in Y\mid\varphi_{q,S}(y)<0\}=\varnothing$. Thus,

$$(\varphi_{q,S,f(\overline{x})}\circ f)(x)=\varphi_{q,S}(f(x)-f(\overline{x}))\geqslant0,\ \forall x\in D.\tag{5}$$

In addition, from $\{t\in\mathbf{R}_+\mid tq\in S\}\subset\{t\in\mathbf{R}\mid tq\in S\}$, it is clear that

$$(\varphi_{q,S,f(\overline{x})}\circ f)(\overline{x})=\varphi_{q,S}(0)=\inf\{t\in\mathbf{R}\mid tq\in S\}\leqslant\varepsilon.$$

Then by (5), we have $(\varphi_{q,S,f(\overline{x})}\circ f)(x)\geqslant(\varphi_{q,S,f(\overline{x})}\circ f)(\overline{x})-\varepsilon$, which implies $\overline{x}\in\mathrm{Amin}(\varphi_{q,S,f(\overline{x})}\circ f,\varepsilon)$.

Conversely, assume that $\overline{x}\in\mathrm{Amin}(\varphi_{q,S,f(\overline{x})}\circ f,\varepsilon)$ and $\overline{x}\notin\mathrm{WAE}(f,D,\mathrm{icr}S)$. Then there exists $\hat{x}\in D$ such that $f(\hat{x})-f(\overline{x})\in-\mathrm{icr}S$. Again by Theorem 2(i), for any $c\in\mathbf{R}$, $\varphi_{q,S}(cq+f(\hat{x})-f(\overline{x}))<c$. Then by letting $c=0$, we have $\varphi_{q,S}(f(\hat{x})-f(\overline{x}))<0$. On the other hand, since $\overline{x}\in\mathrm{Amin}(\varphi_{q,S,f(\overline{x})}\circ f,\varepsilon)$, it follows that

$$\varphi_{q,S}(f(\hat{x})-f(\overline{x}))\geqslant\varphi_{q,S}(f(\overline{x})-f(\overline{x}))-\varepsilon=\varphi_{q,S}(0)-\varepsilon=0.$$

This is a contradiction.

Theorem 6　Let S satisfy Assumption B via relative topological interior, $0\neq q\in S$, $0\notin \mathrm{ri}S$ and $\varepsilon=\inf\{t\in \mathbf{R}_+ \mid tq\in S\}$. Then $\bar{x}\in \mathrm{WAE}(f,D,\mathrm{ri}S)\Leftrightarrow \bar{x}\in \mathrm{Amin}(\varphi_{q,S,f(\bar{x})}\circ f,\varepsilon)$.

Proof　By Theorem 3 (i) and the proof of Theorem 5, the result is clear.

Corollary 8　Let Y be a real topological linear space, $K\subset Y$ be a convex cone. Assume that E is an improvement set with respect to K, $q\in \mathrm{int}(E\cap K)$ and $\varepsilon=\inf\{t\in \mathbf{R}_+ \mid tq\in E\}$. Then

$$\bar{x}\in \mathrm{WAE}(f,D,\mathrm{int}E)\Leftrightarrow \bar{x}\in \mathrm{Amin}(\varphi_{q,E,f(\bar{x})}\circ f,\varepsilon).$$

Proof　We take $S=E$ in Theorem 6 and by virtue of Theorem 3.1 in [20], the result is at hand.

Corollary 9　Let Y be a real topological linear space. Assume that $C\subset Y$ be a proper star-shaped co-radiant set, $0\neq q\in \mathrm{int}(\mathrm{kern}C)$, $\varepsilon>0$ and $\beta=\inf\{t\in \mathbf{R}_+ \mid tq\in C(\varepsilon)\}$. Then

$$\bar{x}\in \mathrm{WAE}(f,D,\mathrm{int}C(\varepsilon))\Leftrightarrow \bar{x}\in \mathrm{Amin}(\varphi_{q,C(\varepsilon),f(\bar{x})}\circ f,\beta).$$

Proof　We take $S=C(\varepsilon)$ in Theorem 6 and then by virtue of Lemma 5.2(iii) in [15], the result is at hand.

Theorem 7　Let S satisfy Assumption B via quasi relative interior, $0\neq q\in S$ and $\varepsilon=\inf\{t\in \mathbf{R}_+ \mid tq\in S\}$. Then $\bar{x}\in \mathrm{WAE}(f,D,\mathrm{qri}S)\Rightarrow \bar{x}\in \mathrm{Amin}(\varphi_{q,S,f(\bar{x})}\circ f,\varepsilon)$.

Proof　By Theorem 4(i) and the proof of Theorem 5, the result is at hand.

Corollary 10　Let S satisfy Assumption B via quasi interior, $0\neq q\in S$ and $\varepsilon=\inf\{t\in \mathbf{R}_+ \mid tq\in S\}$. Then $\bar{x}\in \mathrm{WAE}(f,D,\mathrm{qi}S)\Rightarrow \bar{x}\in \mathrm{Amin}(\varphi_{q,S,f(\bar{x})}\circ f,\varepsilon)$.

Proof　By Corollary 5 (i) and the proof of Theorem 5, the result is at hand.

References:

[1] LUC D T. Theory of vector optimization[M]. Berlin: Springer-Verlag, 1989.

[2] CHEN G Y, HUANG X X, YANG X Q. Vector optimization: set-valued and variational analysis[M]. Berlin: Springer-Verlag, 2005.

[3] ZHAO K Q, CHEN G Y, YANG X M. Approximate proper efficiency in vector optimization[J]. Optimization, 2015, 64(8): 1777-1793.

[4] QIU J H. Dual characterization and scalarization for Benson proper efficiency[J]. SIAM Journal on Optimization, 2008, 19(1): 144-162.

[5] YANG X M, LI D, WANG S Y. Near-subconvexlikeness in vector optimization with set-valued functions[J]. Journal of Optimization Theory and Applications, 2001, 110(2): 413-427.

[6] YANG X M, YANG X Q, CHEN G Y. Theorems of the alternative and optimization with set-valued maps[J]. Journal of Optimization Theory and Applications, 2000, 107(3): 627-640.

[7] HOLMES R B. Geometric functional analysis and its applications, graduate texts in mathematics[M]. New York: Springer-Verlag, 1975.

[8] ADÁN M, NOVOV. Weak efficiency in vector optimization using a closure of algebraic type under a closure of algebraic type under cone-convexlikeness[J]. European Journal of Operational Research, 2003, 149(3): 641-653.

[9] LIMBER M A, GOODRICH R K. Quasi interiors, lagrangemultipliers, and L^p spectral estimation with lattice bounds[J]. Journal of Optimization Theory and Applications, 1993, 78(1): 143-161.

[10] BORWEIN J M, LEWIS A S. Partially finite convex programming, Part I: quasi relative interiors and duality theory[J]. Mathematical Programming, 1992, 57(1): 15-48.

[11] ZĂLINESCU C. On three open problems related to quasi relative interior[J]. Journal of Convex Analysis, 2015, 22(2): 641-645.

[12] BAO T Q, MORDUKHOVICH B S. Relative Pareto minimizers for multiobjectiveproblems: existence and optimality conditions[J]. Mathematical Programming, 2010, 122(2): 301-347.

[13] ZHOU Z A, YANG X M. Optimality conditions of generalized subconvexlike set-valued optimization problems based on the quasi-relative interior[J]. Journal of Optimization Theory and Applications, 2011, 150(2): 327-340.

[14] GÖPFERT A, TAMMER C, RIAHI H, et al. Variational methods in partially ordered spaces[M]. New York: Springer-Verlag, 2003.

[15] GUTIÉRREZ C, JIMÉNEZ B, NOVO V. A unified approach and optimality conditions for approximate solutions of vector optimization problems[J]. SIAM Journal on Optimization, 2006, 17(3): 688-710.

[16] GAO Y, YANG X M, TEO K L. Optimality conditions for approximate solutions of vector optimization problems[J]. Journal of Industrial and Management Optimization,

2011,17(2):483-496.

[17] CHICCO M, MIGNANEGO F, PUSILLO L, et al. Vector optimization problem via improvement sets[J]. Journal of Optimization Theory and Applications, 2011, 150(3): 516-529.

[18] ZHAO K Q, XIA Y M, YANG X M. Nonlinears calarization characterizations of *E*-efficiency in vector optimization[J]. Taiwanese Journal of Mathematics, 2015, 19(2): 455-466.

[19] GUTIÉRREZ C, JIMÉNEZ B, NOVO V. Improvement setsand vector optimization[J]. European Journal of Operational Research, 2012, 223(2): 304-311.

[20] ZHAO K Q, YANG X M. *E*-Benson proper efficiency in vector optimization[J]. Optimization, 2015, 64(4): 739-752.

[21] XIA Y M, ZHANG W L, ZHAO K Q. Characterizations of improvement sets via quasi interior and applications in vector optimization[J]. Optimization Letters, 2016, 10(4): 769-780.

[22] FLORES-BAZÁN F, HERNÁNDEZ E. A unified vector optimization problem: complete scalarizations and applications[J]. Optimization, 2011, 60(12): 1399-1419.

运筹学与控制论

基于假设 B 的非凸分离定理及其在向量优化中的应用

赵克全，朱　巧，汪定国

（重庆师范大学 数学科学学院，重庆 401331）

摘要：【目的】非凸分离定理在研究向量优化问题非线性标量化方法中具有十分重要的作用。【方法】利用 Gerstewitz 非线性标量化函数和假定 B 研究了非凸分离定理及其在向量优化中的应用。【结果】利用具有广义内部的假定 B 建立了一些非凸分离定理并讨论了 free disposal 集和 co-radiant 集等意义下的一些特殊情形。作为其应用，获得了向量优化中基于广义内部定义的几类弱有效解的一些非线性标量化结果。此外，也给出了一些例子对主要结果进行解释。【结论】为研究具有空的拓扑内部或空的相对内部甚至是空的相对代数内部的序锥的向量优化问题提供了新的方法。

关键词：非凸分离定理；广义内部；弱有效解；非线性标量化；向量优化

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