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A Class of E-α-Prequasiinvexity and Mathematical Programming*

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Abstract: [**Purposes**] A new class of generalized convex function named E-α-prequasiinvex function is proposed. Properties and applications of it are studied. [**Methods**] Since E-α-prequasiinvexity is the real generalization of E-preinvexity and α-prequasiinvexity, the results are obtained by generalizing properties of E-preinvexity and α-prequasiinvexity. Examples are given to exam the results. [**Findings**] Firstly, the definitions of E-α-invex set and E-α-prequasiinvex function are given. Examples are introduced to exam their existence. Then, some important properties of E-α-prequasiinvex function are introduced. Moreover, with Condition A and Condition C, the sufficient and necessary condition for E-α-prequasiinvex function is obtained. Finally, applications of E-α-prequasiinvexity in multi-objective programming with constraints as well as the problem without constraints are discussed. [**Conclusions**] The properties and applications of E-α-prequasiinvex functions are studied.

Keywords: E-α-preinvexity; E-α-prequasiinvexity; multi-objective programming; α-preinvexity; α-prequasiinvexity
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1 Introduction

Convexity plays an important role in optimization theory as its properties make it an efficient tool to solve practical problems. In this case, many researches have been conducted to generalize convex functions. An important achievement is the introduction of preinvex functions introduced by Hanson and Mond[1]. Another generalization of convex functions is E-convex functions introduced by Youness[2], which generalized the definitions of convex sets and convex functions based the effort of an operator E on the sets. Inspired by Youness's work, Syau and Lee[3] introduced a new class of generalized convex functions namely E-quasiinvex functions, which is the generalization of E-convex functions and quasiinvex functions. Fulga and Preda[4] extended the class of E-convex sets and E-quasiinvex functions to E-invex sets and E-prequasiinvex functions, respectively.

Another extension is the concepts of α-preinvex functions and α-prequaiinvex functions introduced by Noor M and Noor K[5]. Fan and Guo[6] studied the relationships among α-preinvexity, α-invexity and α-monotonicity. Recently, Liu[7] introduced a new class of generalized convex functions named the strongly quasi α-preinvex

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functions. In addition, Wang and Fu[8] studied some important properties of α-preinvex function, followed by Li and Peng's work[9] which generalized Wang and Fu's results to α-prequasiinvexity. Besides, applications of convex and generalized convex functions are considered by many researchers[6,10-13,17]. Inspired by the earlier works, in this paper, we introduce a new class of generalized convex function named E-α-prequasiinvex functions. It is a true generation of α-preinvex functions and E-preinvex functions. Some important properties and applications in nonlinear programming and multi-objective programming are studied.

2 Definitions and preliminaries

Let H be a real Hilbert space and K be a nonempty subset of H. Let $\alpha: K\times K\to \mathbf{R}\backslash\{0\}$ be a real-valued function, and $\eta: K\times K\to H$ be a vector-valued mapping.

Firstly, we recall the definition of α-invex set.

Definition 1[5] A set $K\subseteq H$ is said to be α-invex with respect to the given mappings $\alpha: K\times K\to \mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to H$, if for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, $\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y})\in K$.

Then, we recall the concepts of α-preinvex function and α-prequasiinvex function.

Definition 2[5] Let $K\subseteq H$ be an α-invex set with respect to the given mappings $\alpha: K\times K\to \mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to H$. A function $f: K\to H$ is said to be a α-preinvex function with respect to $\alpha(\boldsymbol{x},\boldsymbol{y})$ and $\eta(\boldsymbol{x},\boldsymbol{y})$, if for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, $f(\boldsymbol{y})+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\leqslant\lambda f(\boldsymbol{x})+(1-\lambda)f(\boldsymbol{y})$.

Remark 1 A α-preqinvex function on K is preinvex when $\alpha(\boldsymbol{x},\boldsymbol{y})=1$.

Definition 3[5] Let $K\subseteq H$ be an α-invex set with respect to the given mappings $\alpha: K\times K\to \mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to H$. A function $f: K\to H$ is said to be an α-prequasiinvex function with respect to $\alpha(\boldsymbol{x},\boldsymbol{y})$ and $\eta(\boldsymbol{x},\boldsymbol{y})$, if for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, $f(\boldsymbol{y}+\lambda\alpha(\boldsymbol{x},\boldsymbol{y})\eta(\boldsymbol{x},\boldsymbol{y}))\leqslant\max\{f(\boldsymbol{x}),f(\boldsymbol{y})\}$.

In the following section, let $\alpha: K\times K\to \mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to H$ be two fixed mappings. Now, we give the definition of E-α-invex set.

Definition 4 A set $K\subseteq H$ is said to be E-α-invex with respect to the given mappings $\alpha: K\times K\to \mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to H$, if for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, there exists mapping $E: H\to H$ such that

$$E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))\in K.$$

The following example shows the existence of E-α-invex set.

Example 1 Let K be $[-2,-1]\cup[1,2]$. Suppose that mapping E is defined by $E(x)=\begin{cases} x^2, |x|\leqslant\sqrt{2}, \\ -1, |x|>\sqrt{2}. \end{cases}$

Analysis By definition 4, we can prove that the given set K is E-α-invex with respect to $\alpha(x,y)=\begin{cases} xy, xy\neq 0 \\ 1, xy=0 \end{cases}$ and η defined by $\eta(x,y)=\begin{cases} \dfrac{x-y}{xy}, xy>0 \\ \dfrac{-2-y}{xy}, x>0, y<0 \\ \dfrac{1-y}{xy}, x<0, y>0 \\ 1, xy=0 \end{cases}$.

Lemma 1 If $K\subseteq H$ is an E-α-invex set, then $E(K)\subseteq K$.

Proof Since $K\subseteq H$ is an E-α-invex set, for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, we have

$$E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))\in K.$$

Taking $\lambda=0$, we have $E(\boldsymbol{y})\in K$, $\forall\, \boldsymbol{y}\in K$, which implies that $E(K)\subseteq K$.

Let $E: H\to H$ be a fixed mapping, we now give definitions of E-α-preinvex and E-α-prequasiinvex functions.

Definition 5 Let $K\subseteq H$ be an E-α-invex set with respect to the given mappings $\alpha: K\times K\to \mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to H$. A function $f: K\to H$ is said to be an E-α-preinvex function with respect to $\alpha(\boldsymbol{x},\boldsymbol{y})$ and $\eta(\boldsymbol{x},\boldsymbol{y})$, if for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, $f(E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\leqslant\lambda f(E(\boldsymbol{x}))+(1-\lambda)f(E(\boldsymbol{y}))$.

Definition 6 Let $K \subseteq H$ be an E-α-invex set with respect to the given mappings $\alpha: K \times K \to \mathbf{R} \setminus \{0\}$ and $\eta: K \times K \to H$. A function $f: K \to H$ is said to be an E-α-prequasiinvex function with respect to $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and $\eta(\boldsymbol{x}, \boldsymbol{y})$, if for all $\boldsymbol{x}, \boldsymbol{y} \in K$ and $\lambda \in [0,1]$, $f(E(\boldsymbol{y}) + \lambda\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y}))) \leqslant \max\{f(E(\boldsymbol{x})), f(E(\boldsymbol{y}))\}$.

Now, we give Example 2 to illustrate the existence of E-α-prequasiinvex functions.

Example 2 Let $K := [-1, 0]$, $F(x) := \log_2(1-x)$, $\alpha(x,y) \equiv 1$ and $\eta(x,y) := -\frac{y}{4}$. It is easy to prove that the given set K is E-α-invex with respect to $\alpha(x,y)$ and $\eta(x,y)$ when the mapping $E: \mathbf{R} \to \mathbf{R}$ is defined by

$$E(x) = \begin{cases} x^2 - 1, -1 \leqslant x \leqslant -\frac{1}{2} \\ -\frac{3}{4}, -\frac{1}{2} \leqslant x \leqslant 0 \end{cases}.$$

Analysis Now, we prove the E-α-prequasiinvexity of F. For all $x, y \in K$ and $\lambda \in [0,1]$, we have

$$F(E(y) + \lambda\alpha(E(x), E(y))\eta(E(x), E(y))) = F\left(E(y) - \frac{\lambda}{4}E(y)\right) = \log_2\left(1 - E(y) + \frac{\lambda}{4}E(y)\right) \leqslant \log_2(1 - E(y)) \leqslant \max\{F(E(x)), F(E(y))\}.$$

Therefore, F is E-α-prequasiinvex function.

Remark 2 The class of E-α-prequasiinvex function is a true generation of α-preinvex function in [5] and E-preinvex function in [4].

3 Some properties and characterizations

In this section, we show some important properties of E-α-prequasiinvex functions.

Theorem 1 Let $K \subseteq H$ be an E-α-invex set with respect to $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and $\eta(\boldsymbol{x}, \boldsymbol{y})$, then any nonnegative linear combination of E-α-preinvex functions with respect to $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and $\eta(\boldsymbol{x}, \boldsymbol{y})$ is still E-α-preinvex with respect to the same $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and $\eta(\boldsymbol{x}, \boldsymbol{y})$.

Proof Suppose that $f_i, i = 1, 2, 3, \cdots, n$ are E-α-preinvex functions on K, and $F(\boldsymbol{x})$ is defined by $F(\boldsymbol{x}) = \sum_{i=1}^{n} \beta_i f_i(\boldsymbol{x}), \beta_i \geqslant 0, \forall i \in \{1, 2, 3, \cdots, n\}$.

For all $\boldsymbol{x}, \boldsymbol{y} \in K, \beta_i \geqslant 0$ and $\lambda \in [0,1]$, we have

$$F(E(\boldsymbol{y}) + \lambda\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y}))) = \sum_{i=1}^{n} \beta_i f_i(E(\boldsymbol{y}) + \lambda\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y}))) \leqslant \sum_{i=1}^{n} (\lambda\beta_i f_i(E(\boldsymbol{x})) + (1-\lambda)\beta_i f_i(E(\boldsymbol{y}))) = \lambda F(E(\boldsymbol{x})) + (1-\lambda)F(E(\boldsymbol{y})).$$

Theorem 2 Let $K \subseteq H$ be an E-α-invex set with respect to $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and $\eta(\boldsymbol{x}, \boldsymbol{y})$. Suppose that $f: K \to H$ is E-α-preinvex with respect to $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and $\eta(\boldsymbol{x}, \boldsymbol{y})$, and $g: H \to H$ is a nondecreasing convex function, then $g \circ f: K \to H$ is E-α-preinvex with respect to $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and $\eta(\boldsymbol{x}, \boldsymbol{y})$.

Proof Suppose that $f(\boldsymbol{x})$ is E-α-preinvex and $g(\boldsymbol{x})$ is a nondecreasing convex function on K. For all $\boldsymbol{x}, \boldsymbol{y} \in K$ and $\lambda \in [0,1]$, we have

$$(g \circ f)(E(\boldsymbol{y}) + \lambda\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y}))) = g(f(E(\boldsymbol{y}) + \lambda\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y})))) \leqslant g(\lambda f(E(\boldsymbol{x})) + (1-\lambda)f(E(\boldsymbol{y}))) \leqslant \lambda(g \circ f)(E(\boldsymbol{x})) + (1-\lambda)(g \circ f)(E(\boldsymbol{y})).$$

Inspired by Lemma 4.1 in [7], we obtain the following theorem.

Theorem 3 Let $K \subseteq H$ be an E-α-invex set with respect to $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and $\eta(\boldsymbol{x}, \boldsymbol{y})$. Let the mapping $E: H \to H$ be surjective. If for all $\boldsymbol{x}, \boldsymbol{y} \in K$ and $\lambda \in [0,1]$,

$$\eta(E(\boldsymbol{y}), E(\boldsymbol{y}) + \lambda\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y}))) = -\lambda\eta(E(\boldsymbol{x}), E(\boldsymbol{y})), \alpha(E(\boldsymbol{x}), E(\boldsymbol{y})) = \alpha(E(\boldsymbol{y}), E(\boldsymbol{y}) + \lambda\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y}))).$$

then for all $\lambda_1, \lambda_2 \in [0,1]$, we have

$$\eta(E(\boldsymbol{y}) + \lambda_1\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y})), E(\boldsymbol{y}) + \lambda_2\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y}))) = (\lambda_1 - \lambda_2)\eta(E(\boldsymbol{x}), E(\boldsymbol{y})),$$

and $\alpha(E(\boldsymbol{y}) + \lambda_1\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y})), E(\boldsymbol{y}) + \lambda_2\alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))\eta(E(\boldsymbol{x}), E(\boldsymbol{y}))) = \alpha(E(\boldsymbol{x}), E(\boldsymbol{y}))$.

Proof According to above assumption, we have

$$\eta(E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})),E(\boldsymbol{y})+\lambda_2\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=$$
$$\eta(E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})),E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))+$$
$$(\lambda_2-\lambda_1)\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=\eta(E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})),E(\boldsymbol{y})+$$
$$\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))+\left(\frac{\lambda_1-\lambda_2}{\lambda_1}\right)\alpha(E(\boldsymbol{y}),E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\eta(E(\boldsymbol{y}),E(\boldsymbol{y})+$$
$$\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))))=-\frac{\lambda_1-\lambda_2}{\lambda_1}\eta(E(\boldsymbol{y}),E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=$$
$$(\lambda_1-\lambda_2)\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))$$

and

$$\alpha(E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})),E(\boldsymbol{y})+\lambda_2\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=$$
$$\alpha(E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})),E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))+$$
$$\frac{\lambda_1-\lambda_2}{\lambda_1}\alpha(E(\boldsymbol{y}),E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\eta(E(\boldsymbol{y}),E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))))=$$
$$\alpha(E(\boldsymbol{y}),E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=\alpha(E(\boldsymbol{x}),E(\boldsymbol{y})).$$

Motivated by Condition A and Condition C in [5], we introduce following conditions.

Condition A Let $K\subseteq H$ be an E-α-invex set . $F(\boldsymbol{x})$ satisfies Condition A, if for all $\boldsymbol{x},\boldsymbol{y}\in K$,

$$F(E(\boldsymbol{y})+\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\leqslant F(E(\boldsymbol{x})).$$

Condition C Let $K\subseteq H$ be an E-α-invex set. Mappings $\eta:K\times K\to H$ and $\alpha:K\times K\to\mathbf{R}\backslash\{0\}$ satisfy Condition C, if for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$,

$$\eta(E(\boldsymbol{y}),E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=-\lambda\eta(E(\boldsymbol{x}),E(\boldsymbol{y})),$$
$$\eta(E(\boldsymbol{x}),E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=(1-\lambda)\eta(E(\boldsymbol{x}),E(\boldsymbol{y})).$$

Now we give the necessary and sufficient condition of E-α-prequasiinvex functions.

Theorem 4 Let $K\subseteq H$ be an E-α-invex set with respect to $\alpha(\boldsymbol{x},\boldsymbol{y})$ and $\eta(\boldsymbol{x},\boldsymbol{y})$. Let mapping $E:H\to H$ be surjective. Suppose that Condition A and Condition C hold, and for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, $\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))=\alpha(E(\boldsymbol{y}),E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))$. Then, $F:H\to K$ is E-α-prequasiconvex with respect to $\alpha(\boldsymbol{x},\boldsymbol{y})$ and $\eta(\boldsymbol{x},\boldsymbol{y})$, if and only if for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$,

$$g(\lambda)=F(E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))$$

is quasiconvex.

Proof "$\Leftarrow$". If for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, $g(\lambda)=F(E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))$ is quasiconvex, we have

$$F(E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=g(\lambda)=g(1\cdot\lambda+0\cdot(1-\lambda))\leqslant$$
$$\max\{g(1),g(0)\}=\max\{F(E(\boldsymbol{y})+\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))),F(E(\boldsymbol{y}))\}.$$

Since Condition A holds, we consequently have

$$F(E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\leqslant\max\{F(E(\boldsymbol{x})),F(E(\boldsymbol{y}))\}.$$

Therefore, $F(\boldsymbol{x})$ is E-α-prequasiconvex.

"$\Rightarrow$". If $F(\boldsymbol{x})$ is E-α-prequasiconvex, for all $\boldsymbol{x},\boldsymbol{y}\in K$ and $\lambda\in[0,1]$, we have

$$F(E(\boldsymbol{y})+\lambda\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\leqslant\max\{F(E(\boldsymbol{x})),F(E(\boldsymbol{y}))\}.$$

For all $\lambda_1,\lambda_2,\beta\in[0,1]$,

(a) If $\lambda_1\neq\lambda_2$, without loss of generality, supposing $\lambda_2<\lambda_1$, we have

$$g(\beta\lambda_1+(1-\beta)\lambda_2)=F(E(\boldsymbol{y})+(\beta\lambda_1+(1-\beta)\lambda_2)\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=$$
$$F(E(\boldsymbol{y})+\lambda_2\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))+\beta(\lambda_1-\lambda_2)\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=$$
$$F(E(\boldsymbol{y})+\lambda_2\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y}))+\beta\alpha(E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})),$$
$$E(\boldsymbol{y})+\lambda_2\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\eta(E(\boldsymbol{y})+\lambda_1\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})),$$
$$E(\boldsymbol{y})+\lambda_2\alpha(E(\boldsymbol{x}),E(\boldsymbol{y})\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\leqslant$$
$$\max\{F(E(\boldsymbol{y})+\lambda_1\alpha(\mathrm{E}(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))\},F(E(\boldsymbol{y})+\lambda_2\alpha(E(\boldsymbol{x}),E(\boldsymbol{y}))\eta(E(\boldsymbol{x}),E(\boldsymbol{y})))=$$
$$\max\{g(\lambda_1,g(\lambda_2))\}$$

which implies that $g(\lambda)$ is quasiconvex.

(b) If $\lambda_1=\lambda_2$, we have $g(\beta\lambda_1+(1-\beta)\lambda_2)=g(\lambda_1)=g(\lambda_2)\leqslant\max\{g(\lambda_1),g(\lambda_2)\}$. In this case, $g(\lambda)$ is quasiconvex.

Theorem 5　Let $K\subseteq H$ be an E-α-invex set and $f(\boldsymbol{x})$ be E-α-prequasiinvex with respect to $\alpha(\boldsymbol{x},\boldsymbol{y})$ and $\eta(\boldsymbol{x},\boldsymbol{y})$ on K. If $\boldsymbol{x}^*$ is the local minimizer of f and $E:K\to K$ is surjective, then $\boldsymbol{x}^*$ is a global minimizer of f.

Proof　If $\boldsymbol{x}^*$ is not a global minimum point, there must be $\bar{\boldsymbol{x}}\in K(\bar{\boldsymbol{x}}\neq\boldsymbol{x}^*)$, such that $f(\bar{\boldsymbol{x}})<f(\boldsymbol{x}^*)$. Since $E:K\to K$ is surjective, there exists $\boldsymbol{x}_1,\boldsymbol{x}_2\in K$ such that $\boldsymbol{x}^*=E(\boldsymbol{x}_1)$ and $\bar{\boldsymbol{x}}=E(\boldsymbol{x}_2)$. Also, since $\boldsymbol{x}^*$ is local minimum, there exists $\varepsilon>0$, such that for all $\boldsymbol{x}\in U(\boldsymbol{x}^*,\varepsilon)\backslash\{\boldsymbol{x}^*\}, f(\boldsymbol{x}^*)<f(\boldsymbol{x})$. Since $f(\boldsymbol{x})$ is E-α-prequasiinvex, for $\boldsymbol{x}^*,\bar{\boldsymbol{x}}\in K$ and $\lambda\in[0,1]$, we have

$$f(\boldsymbol{x}^*+\lambda\alpha(\bar{\boldsymbol{x}},\boldsymbol{x}^*)\eta(\bar{\boldsymbol{x}},\boldsymbol{x}^*))=f(E(\boldsymbol{x}_1)+\lambda\alpha(E(\boldsymbol{x}_2),E(\boldsymbol{x}_1))\eta E(\boldsymbol{x}_2),E(\boldsymbol{x}_1)))\leqslant$$
$$\max\{f(E(\boldsymbol{x}_1)),f(E(\boldsymbol{x}_2))\}=\max\{f(\boldsymbol{x}^*),f(\bar{\boldsymbol{x}})\}=f(\boldsymbol{x}^*).$$

We choose $\bar{\lambda}=\min\left\{1,\frac{\varepsilon}{\|\alpha(\bar{\boldsymbol{x}},\boldsymbol{x}^*)\eta(\bar{\boldsymbol{x}},\boldsymbol{x}^*)\|}\right\}$, then for all $\lambda'\in[0,\bar{\lambda}]$ we have $\|\boldsymbol{x}^*+\lambda'\alpha(\bar{\boldsymbol{x}},\boldsymbol{x}^*)\eta(\bar{\boldsymbol{x}},\boldsymbol{x}^*)-\boldsymbol{x}^*\|\leqslant\varepsilon$.

Therefore, there exists at least one $\tilde{\boldsymbol{x}}=\boldsymbol{x}^*+\lambda'\alpha(\bar{\boldsymbol{x}},\boldsymbol{x}^*)\eta(\bar{\boldsymbol{x}},\boldsymbol{x}^*)\in U(\boldsymbol{x}^*,\varepsilon)\backslash\{\boldsymbol{x}^*\}$, such that $f(\tilde{\boldsymbol{x}})\leqslant f(\boldsymbol{x}^*)$, which contradicts the fact that $\boldsymbol{x}^*$ is a local minimizer.

The following example is given to illustrate the obtained result.

Example 3　Let $K:=[0,1]$ and $E(x):=x^2+\frac{1}{4}$. Also, we suppose that $\alpha(x,y)=x$ and $\eta(x,y)=-\frac{y}{x}$.

Let the function f be $f(x)=\begin{cases}-x^2+x+1, 0\leqslant x\leqslant\frac{1}{2}\\ \frac{5}{4}, \frac{1}{2}\leqslant x<\infty\end{cases}$.

Firstly, by virtue of Definition 5, one can observe that $f(x)$ is E-α-prequasiinvex function with respect to $\alpha(x,y)$ and $\eta(x,y)$. Then, for the local minimum point $x=0$ of f, it is clear to see that $x=0$ is also a global minimum point. Therefore, the obtained result (Theorem 5) is applicable.

4 Applications in multi-objective programming

In this section, let $E:\mathbf{R}^n\to\mathbf{R}^n$ be a fixed mapping, we firstly consider the following multi-objective programming problem (MOP1):

$$\begin{aligned}&\min\quad f(\boldsymbol{x})=(f_1(\boldsymbol{x}),f_2(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x})),\\&\text{s. t.}\quad \boldsymbol{x}\in K.\end{aligned}$$

where $K\subseteq\mathbf{R}^n$ is an E-α-invex set with respect to $\alpha:K\times K\to\mathbf{R}\backslash\{0\}$ and $\eta:K\times K\to\mathbf{R}^n$, $f:K\to\mathbf{R}^m$ is a vector-valued function and $f_i:K\to\mathbf{R}, i=1,2,3,\cdots,m$ are E-α-prequasiinvex functions with respect to the same mapping $\alpha:K\times K\to\mathbf{R}\backslash\{0\}$ and $\eta:K\times K\to\mathbf{R}^n$.

Remark 3　Throughout this section, let

$$\mathbf{R}_+^m=\{\boldsymbol{x}\in\mathbf{R}^m\mid\boldsymbol{x}=(x_1,x_2,\cdots,x_m),x_i\geqslant0,i=1,2,3,\cdots,m\},$$
$$\mathbf{R}_{++}^m=\{\boldsymbol{x}\in\mathbf{R}^m\mid\boldsymbol{x}=(x_1,x_2,\cdots,x_m),x_i>0,i=1,2,3,\cdots,m\}.$$

Firstly, we recall definitions of efficient solution and weakly efficient solution.

Definition 7[14]　A point $\bar{\boldsymbol{x}}$ is said to be global efficient solution of (MOP1), if there does not exist any feasible point $\boldsymbol{y}\in K$, such that $f(\boldsymbol{y})\in f(\bar{\boldsymbol{x}})-\mathbf{R}_+^m\backslash\{\boldsymbol{0}\}$.

Similarly, a point $\bar{\boldsymbol{x}}$ is said to be local efficient solution of (MOP1), if there is a neighborhood $N(\bar{\boldsymbol{x}},\varepsilon)$ of $\bar{\boldsymbol{x}}$, such that there does not exist any feasible point $\boldsymbol{y}\in K\cap N(\bar{\boldsymbol{x}},\varepsilon)$, satisfying $f(\boldsymbol{y})\in f(\bar{\boldsymbol{x}})-\mathbf{R}_+^m\backslash\{\boldsymbol{0}\}$.

Definition 8[14]　A point $\bar{\boldsymbol{x}}$ is said to be global weakly efficient solution of (MOP1), if there does not exist any feasible point $\boldsymbol{y}\in K$, such that $f(\boldsymbol{y})\in f(\bar{\boldsymbol{x}})-\mathbf{R}_{++}^m$.

Similarly, a point $\bar{\boldsymbol{x}}$ is said to be local weakly efficient solution of (MOP1), if there is a neighborhood $N(\bar{\boldsymbol{x}},\varepsilon)$ of $\bar{\boldsymbol{x}}$, such that there does not exist any feasible point $\boldsymbol{y}\in K\cap N(\bar{\boldsymbol{x}},\varepsilon)$, satisfying $f(\boldsymbol{y})\in f(\bar{\boldsymbol{x}})-\mathbf{R}_{++}^m$.

Remark 4 Efficient solutions are also weakly efficient solutions, but the converse is not true.

Inspired by Lemma 3.3 in [15], we obtain Lemma 2.

Lemma 2 Let K be an E-α-invexset, and $f_i, i=1,2,3,\cdots,m$ be E-α-prequasiinvex functions with respect to the same mapping $\alpha: K\times K\to\mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to\mathbf{R}^n$. Then exactly one of the following two systems is solvable:

(i) there exists $\boldsymbol{x}\in K$, such that $f_i(\boldsymbol{x})<0, i=1,2,3,\cdots,m$;

(ii) there exists $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_m)\in\mathbf{R}_+^m\backslash\{\mathbf{0}\}$, such that $\sum_{i=1}^m\lambda_i f_i(\boldsymbol{x})\geqslant 0, \forall\, \boldsymbol{x}\in K$.

Proof Suppose the system (i) is solvable, then there exists $\boldsymbol{x}\in K$, such that $\boldsymbol{y}_i=f_i(\boldsymbol{x})<0, i=1,2,3,\cdots,m$. Therefore, for all $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_m)\in\mathbf{R}_+^m\backslash\{\mathbf{0}\}$, we have $\sum_{i=1}^m\lambda_i f_i(\boldsymbol{x})=\sum_{i=1}^m\lambda_i\boldsymbol{y}_i<0$.

In this case, the system (ii) has no solution. Using the same method, it is easy to prove that (i) has no solution when (ii) is solvable. Therefore exactly one of the following two systems is solvable.

Theorem 6 Let $K\subseteq\mathbf{R}^n$ be an E-α-invex set with respect to $\alpha: K\times K\to\mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to\mathbf{R}^n$. Suppose that $f_i, i=1,2,3,\cdots,m$ are E-α-prequasiinvex functions with respect to the same $\alpha: K\times K\to\mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to\mathbf{R}^n$. If $\boldsymbol{x}^*\subseteq K$ is a global weakly efficient solution of (MOP1), then there must exist $\boldsymbol{\lambda}\in\mathbf{R}_+^m\backslash\{\mathbf{0}\}$ such that $\boldsymbol{x}^*$ is an optimal solution of the following scalar problem (P_λ)

$$\begin{aligned}\min\quad & \boldsymbol{\lambda}\boldsymbol{f}(\boldsymbol{x})^{\mathrm{T}},\\ \text{s. t.}\quad & \boldsymbol{x}\in K,\\ & \boldsymbol{\lambda}\in\mathbf{R}_+^m\backslash\{\mathbf{0}\}.\end{aligned}$$

Proof Suppose that $\boldsymbol{x}^*\in K$ is a global weakly efficient solution for (MOP1), then there does not exist any point $\boldsymbol{x}\in K$, such that $f_i(\boldsymbol{x})-f_i(\boldsymbol{x}^*)<0, i=1,2,3,\cdots,m$. According to Lemma 2, we immediately conclude that there exists $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_m)\in\mathbf{R}_+^m\backslash\{\mathbf{0}\}$, such that $\sum_{i=1}^m\lambda_i(f_i(\boldsymbol{x})-f_i(\boldsymbol{x}^*))\geqslant 0, \forall\,\boldsymbol{x}\in K$.

Then we have $\sum_{i=1}^m\lambda_i f_i(\boldsymbol{x})\geqslant\sum_{i=1}^m\lambda_i f_i(\boldsymbol{x}^*), \forall\,\boldsymbol{x}\in K$, $\boldsymbol{x}^*$ is the optimal solution of (P_λ).

In the sequel, assuming that $E:\mathbf{R}^n\to\mathbf{R}^n$ is surjective, we now consider the multi-objective problem (MOP2) with constraints as follows:

$$\begin{aligned}\min\quad & f(\boldsymbol{x})=(f_1(\boldsymbol{x}),f_2(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x})),\\ \text{s. t.}\quad & g(\boldsymbol{x})=(g_1(\boldsymbol{x}),g_2(\boldsymbol{x}),\cdots,g_n(\boldsymbol{x}))\leqslant 0,\\ & \boldsymbol{x}\in E(K).\end{aligned}$$

where $K\subseteq\mathbf{R}^n$ is an E-α-invex set with respect to $\alpha: K\times K\to\mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to\mathbf{R}^n$, $f: K\to\mathbf{R}^m$ and $g: K\to\mathbf{R}^n$ ($n\geqslant m$) are vector-valued functions. $f_i: K\to\mathbf{R}, i=1,2,3,\cdots,m$ and $g_i: K\to\mathbf{R}, i=1,2,3,\cdots,n$ are E-α-prequasiinvex functions with respect to the same mapping $\alpha: K\times K\to\mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to\mathbf{R}^n$.

Definition 9[12] An efficient solution $\bar{\boldsymbol{x}}$ for (MOP2) is said to be properly efficient solution for (MOP2), if there exists a positive scalar q such that for each $i, i=1,2,3,\cdots,m$ and each feasible $\boldsymbol{x}$ satisfying $f_i(\boldsymbol{x})<f_i(\bar{\boldsymbol{x}})$, there exists at least one index $j\neq i$ with $f_j(\boldsymbol{x})>f_j(\bar{\boldsymbol{x}})$ and $\dfrac{f_i(\boldsymbol{x})-f_i(\bar{\boldsymbol{x}})}{f_j(\bar{\boldsymbol{x}})-f_j(\boldsymbol{x})}\leqslant q$.

Remark 5 For a feasible point $\boldsymbol{y}^*=E(\boldsymbol{x}^*)(\boldsymbol{x}^*\in K)$ for (MOP2), we denote $I(\boldsymbol{x}^*)$ by $I(\boldsymbol{x}^*)=\{j\,|\,(g_i\circ E)(\boldsymbol{x}^*)\}=0,\ j\in\{1,2,3,\cdots,n\}\}$.

Inspired by the Farkas Lemma in [16], we give the following lemma.

Lemma 3 Let $K\subseteq\mathbf{R}^n$ be E-α-invex set and $f_i(\boldsymbol{x}), i=1,2,3,\cdots,m, g_j(\boldsymbol{x}), j=1,2,3,\cdots,n$ be E-α-prequasiinvex functions with respect to the same $\alpha: K\times K\to\mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to\mathbf{R}^n$. Let $\boldsymbol{y}^*=E(\boldsymbol{x}^*)(\boldsymbol{x}^*\in K)$ be a feasible solution of (MOP2), then exactly one of the following systems is solvable:

(i) there exists q, such that for all feasible $\boldsymbol{y}=E(\boldsymbol{x})(\boldsymbol{x}\in K), \boldsymbol{y}\neq\boldsymbol{y}^*$, $(E(\boldsymbol{x})-E(\boldsymbol{x}^*))\nabla f_q(E(\boldsymbol{x}^*))<0$, $(E(\boldsymbol{x})-E(\boldsymbol{x}^*))\nabla f_q(E(\boldsymbol{x}^*))\leqslant 0, \forall\, i\neq q$, $(E(\boldsymbol{x})-E(\boldsymbol{x}^*))\nabla f_q(E(\boldsymbol{x}^*))\leqslant 0, \forall\, j\in I(\boldsymbol{x}^*)$;

(ii) there exists positive $\lambda_i, i=1,2,3,\cdots,m$ and $\mu_j \geqslant 0, j\in I(\boldsymbol{x}^*)$, such that

$$\sum_{i=1}^{m}\lambda_i \nabla f_i(E(\boldsymbol{x}^*)) + \sum_{j\in I(\boldsymbol{x}^*)} \mu_j \nabla g_j(E(\boldsymbol{x}^*)) = 0.$$

Proof Suppose (i) is solvable, then for all positive $\lambda_i, i=1,2,3,\cdots$, and $\mu_j \geqslant 0, j\in I(\boldsymbol{x}^*)$, we have

$$(E(\boldsymbol{x}) - E(\boldsymbol{x}^*))(\sum_{i=1}^{m}\lambda_i \nabla f_i(E(\boldsymbol{x}^*)) + \sum_{j\in I(\boldsymbol{x}^*)} \mu_j \nabla g_j(E(\boldsymbol{x}^*))) < 0.$$

In this case, it is easy to see that (ii) has no solution. Using the same method, we can prove that the system (i) has no solution when (ii) is solvable. Therefore, exactly one of the two systems is solvable.

Theorem 7 Let $f_i(\boldsymbol{x}), i=1,2,3,\cdots,m, g_j(\boldsymbol{x}), j=1,2,3,\cdots,n$ be E-α-prequasiinvex functions with respect to the same $\alpha: K\times K\to \mathbf{R}\backslash\{0\}$ and $\eta: K\times K\to \mathbf{R}^n$. Let $\boldsymbol{y}^*$ be a properly efficient solution for (MOP2), and $\boldsymbol{y}'$ be a feasible solution for (MOP2) such that $g_j(\boldsymbol{y}')<0, j=1,2,3,\cdots,n$. If $f_i(\boldsymbol{x})$, $i=1,2,3,\cdots,m$, $g_j(\boldsymbol{x})$, $j=1,2,3,\cdots,n$ satisfy

$$(E(\boldsymbol{x})-E(\boldsymbol{y}))\nabla f_i(E(\boldsymbol{y}))^{\mathrm{T}} \leqslant (f_i(E(\boldsymbol{x}))-f_i(E(\boldsymbol{y}))),\ \forall\, \boldsymbol{x},\boldsymbol{y}\in K, \tag{1}$$

$$(E(\boldsymbol{x})-E(\boldsymbol{y}))\nabla g_j(E(\boldsymbol{y}))^{\mathrm{T}} \leqslant (g_j(E(\boldsymbol{x}))-g_j(E(\boldsymbol{y}))),\ \forall\, \boldsymbol{x},\boldsymbol{y}\in K, \tag{2}$$

then, there exists positive $\lambda_i, i=1,2,3,\cdots,m$ and $\mu_j\geqslant 0, j\in I(\boldsymbol{x}^*)$, such that $\sum_{i=1}^{m}\lambda_i \nabla f_i(\boldsymbol{y}^*) + \sum_{j\in I(\boldsymbol{x}^*)}\mu_j \nabla g_j(\boldsymbol{y}^*) = 0$.

Proof Since $\boldsymbol{y}^*, \boldsymbol{y}'\in E(K)$ and $E(\cdot)$ is surjective, there are $\boldsymbol{x}^*, \boldsymbol{x}'\in K$, such that $\boldsymbol{y}^*=E(\boldsymbol{x}^*)$ and $\boldsymbol{y}'=E(\boldsymbol{x}')$. For all feasible $\boldsymbol{y}=E(\boldsymbol{x}), \boldsymbol{x}\in K$, we suppose that the following system is solvable:

$$(E(\boldsymbol{x})-E(\boldsymbol{x}^*))\nabla f_q(E(\boldsymbol{x}^*))^{\mathrm{T}}<0, \tag{3}$$

$$(E(\boldsymbol{x})-E(\boldsymbol{x}^*))\nabla f_i(E(\boldsymbol{x}^*))^{\mathrm{T}}\leqslant 0,\ \forall\, i\neq q, \tag{4}$$

$$(E(\boldsymbol{x})-E(\boldsymbol{x}^*))\nabla g_j(E(\boldsymbol{x}^*))^{\mathrm{T}}\leqslant 0,\ \forall\, j\in I(\boldsymbol{x}^*). \tag{5}$$

Since there exists $\boldsymbol{y}'=E(\boldsymbol{x}')$ such that $g_j(E(\boldsymbol{x}'))<0$, we have

$$g_j(E(\boldsymbol{x}'))-g_j(E(\boldsymbol{x}^*))=g_j(E(\boldsymbol{x}'))<0,\ \forall\, j\in I(\boldsymbol{x}^*).$$

According to (2), we have

$$(E(\boldsymbol{x}')-E(\boldsymbol{x}^*))\nabla g_j(E(\boldsymbol{x}^*))^{\mathrm{T}}<0,\ \forall\, j\in I(\boldsymbol{x}^*). \tag{6}$$

By virtue of (5) and (6), for any positive ρ, we have

$$[(E(\boldsymbol{x})-E(\boldsymbol{x}^*))+\rho(E(\boldsymbol{x}')-E(\boldsymbol{x}^*))]\nabla g_j(E(\boldsymbol{x}^*))^{\mathrm{T}}<0,\ \forall\, j\in I(\boldsymbol{x}^*).$$

Therefore, for any sufficiently small positive λ, we have

$$g_j(E(\boldsymbol{x}^*)+\lambda[(E(\boldsymbol{x})-E(\boldsymbol{x}^*))+\rho(E(\boldsymbol{x}')-E(\boldsymbol{x}^*))])<g_j(E(\boldsymbol{x}^*))=0,\ \forall\, j\in I(\boldsymbol{x}^*).$$

For $k\notin I(\boldsymbol{x}^*)$, since $g_k(E(\boldsymbol{x}^*))<0$, we have $g_k(E(\boldsymbol{x}^*)+\lambda((E(\boldsymbol{x})-E(\boldsymbol{x}^*)+\rho(E(\boldsymbol{x}')-E(\boldsymbol{x}^*))))\leqslant 0,\ \forall\, k\notin I(\boldsymbol{x}^*)$. Therefore $E(\boldsymbol{x}^*)+\lambda((E(\boldsymbol{x})-E(\boldsymbol{x}^*))+\rho(E(\boldsymbol{x}')-E(\boldsymbol{x}^*)))$ is a feasible solution for (MOP2).

Using the same method, we have $f_q(E(\boldsymbol{x}^*)+\lambda[(E(\boldsymbol{x})-E(\boldsymbol{x}^*))+\rho(E(\boldsymbol{x}')-E(\boldsymbol{x}^*))])<f_q(E(\boldsymbol{x}^*))$. Because of the proper efficiency of $\boldsymbol{y}^*=E(\boldsymbol{x}^*)$, there exists at least one $i\neq q$ such that

$$f_i(E(\boldsymbol{x}^*)+\lambda[(E(\boldsymbol{x})-E(\boldsymbol{x}^*))+\rho(E(\boldsymbol{x}')-E(\boldsymbol{x}^*))])>f_i(E(\boldsymbol{x}^*)). \tag{7}$$

Let
$$A(\lambda,\rho)=\frac{f_q(E(\boldsymbol{x}^*))-f_q(E(\boldsymbol{x}^*)+\lambda[(E(\boldsymbol{x})-E(\boldsymbol{x}^*))+\rho(E(\boldsymbol{x}')-E(\boldsymbol{x}^*))])}{\lambda},$$

$$B(\lambda,\rho)=\frac{f_i(E(\boldsymbol{x}^*)+\lambda[(E(\boldsymbol{x})-E(\boldsymbol{x}^*))+\rho(E(\boldsymbol{x}')-E(\boldsymbol{x}^*))])-f_i(E(\boldsymbol{x}^*))}{\lambda}.$$

According to (3), it is easy to see that $A(\lambda,\rho)\to(E(\boldsymbol{x}^*)-E(\boldsymbol{x}))\nabla f_q(E(\boldsymbol{x}^*))^{\mathrm{T}}>0, \lambda\to 0, \rho\to 0$.

Similarly, by (4), we have $B(\lambda,\rho)\to(E(\boldsymbol{x})-E(\boldsymbol{x}^*))\nabla f_i(E(\boldsymbol{x}^*))^{\mathrm{T}}\leqslant 0, \lambda\to 0, \rho\to 0$.

However, by the virtue of (7), we have $B(\lambda,\rho)>0$, which implies the limit of $B(\lambda,\rho)$ is nonnegative when $\lambda\to 0$ and $\rho\to 0$. Then, we find that $\dfrac{A(\lambda,\rho)}{B(\lambda,\rho)}\to\infty, \lambda\to 0, \rho\to 0$, which contradicts to the proper efficiency of $\boldsymbol{y}^*=E(\boldsymbol{x}^*)$.

Therefore, the given system is not solvable. According to Lemma 3 we have

$$\sum_{i=1}^{m}\lambda_i \nabla f_i(\boldsymbol{y}^*)+\sum_{j\in I(\boldsymbol{x}^*)}\mu_j \nabla g_j(\boldsymbol{y}^*)=0.$$

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运筹学与控制论

一类 E-α-预不变拟凸性与数学规划

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摘要：【目的】提出一类新的广义凸函数 E-α-预不变拟凸函数，并研究它的性质和应用。【方法】由于 E-α-预不变拟凸性是 E-预不变凸性和 α-预不变拟凸性的真推广，将 E-预不变凸性和 α-预不变拟凸性推广可以得到结果，并举例验证。【结果】首先，给出了 E-α-不变凸集和 E-α-预不变拟凸函数的定义，给出实例说明其存在性。然后，给出了 E-α-预不变拟凸函数的几个重要性质，并借助条件 A 和条件 C 获得了 E-α-预不变拟凸函数的等价刻画。最后，讨论了 E-α-预不变拟凸性分别在无约束与约束多目标规划问题中的应用。【结论】研究了 E-α--预不变拟凸函数的性质和应用。

关键词：E-α-预不变凸性；E-α-预不变拟凸性；多目标规划问题；α-预不变凸性；α-预不变拟凸性

（责任编辑　黄　颖）