

二维各向异性位势问题的改进插值型边界无单元法*

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摘要:【目的】把边界积分方程方法和基于非奇异权函数的改进移动最小二乘插值法相结合,建立数值求解二维各向异性位势问题的改进插值型边界无单元法。【方法】在改进移动最小二乘插值法的基础上,讨论了非奇异权函数的改进移动最小二乘插值法,它的形函数满足 Kronecker δ 函数的性质,因此可以直接施加边界条件。【结果】数值算例表明该方法求解二维各向异性位势问题是有效和可行的。【结论】与边界元方法相比,该方法精度和收敛性更好。

关键词:各向异性;位势问题;非奇异权函数的改进移动最小二乘插值法;改进插值型边界无单元法

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无论晶体、木材等天然材料,还是纤维增强材料、层合板壳等人造材料,它们都具有热传导系数等随方向而改变的特性。该特性包括各向异性和正交各向异性。各向异性材料由于在结构与性能方面具有众多优越性,因此在工程中应用日趋广泛。目前数值求解各向异性位势问题的方法有边界元法^[1-2]及其改进算法^[3],如正则化边界元方法^[4]和虚边界元法^[5]等。

无网格方法^[6-8]是求解偏微分方程边值问题的一类新的数值方法。移动最小二乘法是一种构造无网格方法形函数的主要方法^[9],缺点是法方程组有时是奇异的或者病态的。程玉民等人^[10-11]利用正交基函数对移动最小二乘法进行了改进,提出了改进的移动最小二乘法,并把改进的移动最小二乘法和边界积分方程方法相结合建立了边界无单元法。边界无单元法求解了大量边值问题,但是由于改进的移动最小二乘法构造的形函数不满足插值性质,因此直接施加边界条件会降低计算精度。任红萍等人^[12-13]利用奇异权函数,建立了改进移动最小二乘插值法。改进移动最小二乘插值法的缺点是权函数具有奇异性,为了克服这个缺点,王聚丰等人^[14-15]提出了基于非奇异权函数的改进移动最小二乘插值法,并把它和边界积分方程方法相结合,建立了改进插值型边界无单元法。和边界无单元法相比,改进插值型边界无单元法可以直接施加边界条件,并且具有更高的计算效率。

改进插值型边界无单元法求解了二维 Laplace 方程边值问题,这是一种特殊的正交各向同性问题。本文将建立各向异性问题的改进插值型边界无单元法,采用非奇异权函数,该方法不仅有较高的精度,而且直接施加边界条件更容易。数值算例表明,本文方法比边界元法有更快的收敛速度和更高的精度。

1 基于非奇异权函数的改进移动最小二乘插值法

为了建立函数 $v(\mathbf{x})$ 的逼近函数,定义函数 $\tilde{v}(\mathbf{x}, \bar{\mathbf{x}})$ 为:

$$\tilde{v}(\mathbf{x}, \bar{\mathbf{x}}) = v(\bar{\mathbf{x}}) - \sum_{j=1}^n q(\mathbf{x}, \mathbf{x}_j) v(\mathbf{x}_j) = v(\bar{\mathbf{x}}) - \mathbf{q}(\mathbf{x}) \mathbf{v}. \quad (1)$$

其中 $\mathbf{x}_j, j=1, \dots, n$ 为节点, n 为节点总数, $\mathbf{q}(\mathbf{x}) = (q(\mathbf{x}, \mathbf{x}_1), \dots, q(\mathbf{x}, \mathbf{x}_n))$, $q(\mathbf{x}, \mathbf{x}_j) = \frac{\xi(\mathbf{x}, \mathbf{x}_j)}{\sum_{j=1}^n \xi(\mathbf{x}, \mathbf{x}_j)}$, $\xi(\mathbf{x}, \mathbf{x}_j) =$

$$\prod_{k=1}^n \frac{\mathbf{x} - \mathbf{x}_j}{\mathbf{x}_k - \mathbf{x}_j}, k \neq j, \mathbf{v} = (v(\mathbf{x}_1), v(\mathbf{x}_2), \dots, v(\mathbf{x}_n))^T.$$

定义局部逼近函数 $\tilde{v}_h(\mathbf{x}, \bar{\mathbf{x}})$ 为:

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$$\tilde{v}_h(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{j=1}^m g_j(\mathbf{x}, \bar{\mathbf{x}}) a_j(\mathbf{x}) = \mathbf{g}(\mathbf{x}, \bar{\mathbf{x}}) \mathbf{a}(\mathbf{x}), \quad (2)$$

其中 $\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), \dots, a_m(\mathbf{x}))^\top$, $\mathbf{g}(\mathbf{x}, \bar{\mathbf{x}}) = (g_1(\mathbf{x}, \bar{\mathbf{x}}), \dots, g_m(\mathbf{x}, \bar{\mathbf{x}}))$, $g_i(\mathbf{x}, \bar{\mathbf{x}}) = p_i(\bar{\mathbf{x}}) - \sum_{k=1}^n q(\mathbf{x}, \mathbf{x}_k) p_i(\mathbf{x}_k)$ 。定义泛函, 有 $J = \sum_{i=1}^n \omega_i(\mathbf{x}) [\tilde{v}_h(\mathbf{x}, \bar{\mathbf{x}}) - \tilde{v}(\mathbf{x}, \mathbf{x}_i)]^2 = \sum_{i=1}^n \omega_i(\mathbf{x}) \{ \mathbf{g}(\mathbf{x}, \bar{\mathbf{x}}) \mathbf{a}(\mathbf{x}) - [\mathbf{e}_i - \mathbf{q}(\mathbf{x})] \mathbf{v} \}^2$ 。其中 $\omega_i(\mathbf{x})$ 为非奇异的权函数。为了得到 $\mathbf{a}(\mathbf{x})$, 令 $\frac{\partial J}{\partial a_j(\mathbf{x})} = 0$, 得:

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \mathbf{v}, \quad (3)$$

其中 $\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}), \dots, a_m(\mathbf{x}))^\top$, $\mathbf{A}(\mathbf{x}) = \sum_{i=1}^n \omega_i(\mathbf{x}) \mathbf{g}(\mathbf{x}, \mathbf{x}_i) \mathbf{g}^\top(\mathbf{x}, \mathbf{x}_i)$, $\mathbf{B}(\mathbf{x}) = \sum_{i=1}^n \omega_i(\mathbf{x}) \mathbf{g}(\mathbf{x}, \mathbf{x}_i) [\mathbf{e}_i - \mathbf{q}(\mathbf{x})]$, $\mathbf{v} = (v(\mathbf{x}_1), v(\mathbf{x}_2), \dots, v(\mathbf{x}_n))^\top$, 其中 \mathbf{e}_i 是第 i 个分量等于 1 的 n 维单位行向量。为了得到局部逼近函数 $\tilde{v}_h(\mathbf{x}, \bar{\mathbf{x}})$, 把 (3) 式代入 (2) 式, 得:

$$\tilde{v}_h(\mathbf{x}, \bar{\mathbf{x}}) = \mathbf{g}(\mathbf{x}, \bar{\mathbf{x}}) \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \mathbf{v}. \quad (4)$$

由 (1), (4) 式, 并令 $\bar{\mathbf{x}} = \mathbf{x}$, 从而得到函数 $v(\mathbf{x})$ 的逼近函数为:

$$v_h(\mathbf{x}) = \tilde{v}_h(\mathbf{x}, \mathbf{x}) + \sum_{i=1}^n q(\mathbf{x}, \mathbf{x}_i) v(\mathbf{x}_i) = \sum_{i=1}^n \Phi_i(\mathbf{x}) v(\mathbf{x}_i), \quad (5)$$

其中 $\Phi_i(\mathbf{x}) = \sum_{j=1}^m g_j(\mathbf{x}) [\mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x})]_{ji} + q(\mathbf{x}, \mathbf{x}_i)$, $g_j(\mathbf{x}) = p_j(\mathbf{x}) - \sum_{k=1}^n q(\mathbf{x}, \mathbf{x}_k) p_j(\mathbf{x}_k)$ 。

2 二维各向异性位势问题的改进的插值型边界无单元法

2.1 边界积分方程

对二维各向异性位势问题, 控制方程为:

$$k_{11} \frac{\partial^2 u}{\partial x_1^2} + 2k_{12} \frac{\partial^2 u}{\partial x_1 \partial x_2} + k_{22} \frac{\partial^2 u}{\partial x_2^2} = 0, \mathbf{x} = (x_1, x_2) \in \Omega, \quad (6)$$

边界条件为:

$$u(\mathbf{x}) = \bar{u}(\mathbf{x}), \mathbf{x} \in \Gamma_u, \quad (7)$$

$$q(\mathbf{x}) = \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = \bar{q}(\mathbf{x}), \mathbf{x} \in \Gamma_q. \quad (8)$$

其中 Ω 是问题所在的域, Γ 为 Ω 的边界, 且有 $\Gamma = \Gamma_u \cup \Gamma_q$, $\Gamma_u \cap \Gamma_q = \emptyset$ 。 u 表示场点位势, \bar{u} 是边界 Γ_u 上的已知位势, \bar{q} 是边界 Γ_q 上的已知位势梯度, \mathbf{n} 是边界 Γ 的外法线方向, k_{ij} ($i, j = 1, 2$) 是各向异性材料的特性系数。

二维各向异性位势问题的区域内点 \mathbf{x} 的边界积分方程为:

$$u(\mathbf{x}) = \int_{\Gamma} u^*(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) dS_y - \int_{\Gamma} q^*(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) dS_y, \mathbf{x} \in \Omega, \quad (9)$$

其中位势基本解为:

$$u^*(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi\sqrt{|\mathbf{D}|}} \ln \frac{1}{r(\mathbf{x}, \mathbf{y})}. \quad (10)$$

(10) 式中 $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ 分别为场点和源点, 有:

$$|\mathbf{D}| = \begin{vmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{vmatrix} = k_{11}k_{22} - k_{12}^2. \quad (11)$$

令

$$\mathbf{S} = \mathbf{D}^{-1} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}^{-1} = \frac{1}{|\mathbf{D}|} \begin{bmatrix} k_{22} & -k_{12} \\ -k_{12} & k_{11} \end{bmatrix} = \begin{bmatrix} \frac{k_{22}}{|\mathbf{D}|} & -\frac{k_{12}}{|\mathbf{D}|} \\ -\frac{k_{12}}{|\mathbf{D}|} & \frac{k_{11}}{|\mathbf{D}|} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}, \quad (12)$$

则 $r(\mathbf{x}, \mathbf{y})$ 可表示为:

$$r(\mathbf{x}, \mathbf{y}) = \sqrt{s_{11}(x_1 - y_1)^2 + 2s_{12}(x_1 - y_1)(x_2 - y_2) + s_{22}(x_2 - y_2)^2}. \quad (13)$$

对应的外法向矢量方向位势梯度基本解为:

$$q^*(\mathbf{x}, \mathbf{y}) = \left(k_{11} \frac{\partial u^*(\mathbf{x}, \mathbf{y})}{\partial y_1} + k_{12} \frac{\partial u^*(\mathbf{x}, \mathbf{y})}{\partial y_2} \right) \mathbf{n}_{y_1} + \left(k_{12} \frac{\partial u^*(\mathbf{x}, \mathbf{y})}{\partial y_1} + k_{22} \frac{\partial u^*(\mathbf{x}, \mathbf{y})}{\partial y_2} \right) \mathbf{n}_{y_2} =$$

$$\frac{1}{2\pi\sqrt{|\mathbf{D}|}} \left\{ \frac{[(k_{11}s_{11} + k_{12}s_{12})(x_1 - y_1) + (k_{11}s_{12} + k_{12}s_{22})(x_2 - y_2)] \mathbf{n}_{y_1}}{r^2} + \right.$$

$$\left. \frac{[(k_{12}s_{11} + k_{22}s_{12})(x_1 - y_1) + (k_{12}s_{12} + k_{22}s_{22})(x_2 - y_2)] \mathbf{n}_{y_2}}{r^2} \right\}. \quad (14)$$

其中 $\mathbf{n}_{y_1}, \mathbf{n}_{y_2}$ 为边界外法向矢量方向余弦。

把边界条件(7),(8)式代入(9)式中,得:

$$u(\mathbf{x}) = \int_{\Gamma_u} u^*(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) dS_y + \int_{\Gamma_q} u^*(\mathbf{x}, \mathbf{y}) \bar{q}(\mathbf{y}) dS_y -$$

$$\int_{\Gamma_u} \bar{u}(\mathbf{y}) q^*(\mathbf{x}, \mathbf{y}) dS_y - \int_{\Gamma_q} u(\mathbf{y}) q^*(\mathbf{x}, \mathbf{y}) dS_y, \mathbf{x} \in \Omega. \quad (15)$$

从(15)式可以看出,如果要求出位势 $u(\mathbf{x})$,那么首先要求出在 Γ_u 上的未知量 q 和在 Γ_q 上的未知量 u 。

当 $\mathbf{x} \rightarrow \Gamma$ 时,由(9)式可以得到直接边界积分方程的正则形式^[16]:

$$\int_{\Gamma} u^*(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) dS_y - \int_{\Gamma} [u(\mathbf{y}) - u(\mathbf{x})] q^*(\mathbf{x}, \mathbf{y}) dS_y = 0, \mathbf{x} \rightarrow \Gamma. \quad (16)$$

把边界条件(7),(8)式代入(16)式,把含未知量的项移到等号的左边,再把含已知量的项移到等号的右边,得:

$$\int_{\Gamma_u} u^*(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) dS_y - \int_{\Gamma_q} u(\mathbf{y}) q^*(\mathbf{x}, \mathbf{y}) dS_y =$$

$$\int_{\Gamma_u} [\bar{u}(\mathbf{y}) - \bar{u}(\mathbf{x})] q^*(\mathbf{x}, \mathbf{y}) dS_y - \int_{\Gamma_q} [u^*(\mathbf{x}, \mathbf{y}) \bar{q}(\mathbf{y}) + q^*(\mathbf{x}, \mathbf{y}) \bar{u}(\mathbf{x})] dS_y, \mathbf{x} \in \Gamma_u, \quad (17)$$

$$\int_{\Gamma_u} u^*(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) dS_y + \int_{\Gamma_u} q^*(\mathbf{x}, \mathbf{y}) u(\mathbf{x}) dS_y - \int_{\Gamma_q} q^*(\mathbf{x}, \mathbf{y}) [u(\mathbf{y}) - u(\mathbf{x})] dS_y =$$

$$\int_{\Gamma_u} q^*(\mathbf{x}, \mathbf{y}) \bar{u}(\mathbf{y}) dS_y - \int_{\Gamma_q} u^*(\mathbf{x}, \mathbf{y}) \bar{q}(\mathbf{y}) dS_y, \mathbf{x} \in \Gamma_q. \quad (18)$$

2.2 插值未知函数

用 $\{\mathbf{x}_i\}_{i=1}^N$ 表示 N 个边界节点 $\mathbf{x}_i \in \Gamma$ 的集合。设 N_u 个边界节点 $\{\mathbf{x}_i\}_{i=1}^{N_u} \in \Gamma_u, N - N_u$ 个边界节点 $\{\mathbf{x}_i\}_{i=N_u+1}^N \in \Gamma_q$ 。利用非奇异权函数的改进的移动二乘插值法,位势 u 和位势梯度 $q = \partial u / \partial \mathbf{n}$ 被定义为:

$$q(\mathbf{x}) = \sum_{i=1}^{N_u} \Phi_i(\mathbf{x}) q_i, \mathbf{x} \in \Gamma_u \quad (19)$$

$$u(\mathbf{x}) = \sum_{i=N_u+1}^N \Phi_i(\mathbf{x}) u_i, \mathbf{x} \in \Gamma_q. \quad (20)$$

2.3 离散边界积分方程

对于边界节点 $\{\mathbf{x}_k\}_{k=1}^{N_u} \in \Gamma_u$ 时,把(19),(20)式代入(17)式,得:

$$\sum_{i=1}^{N_u} q_i \int_{\Gamma_u} u^*(\mathbf{x}_k, \mathbf{y}) \Phi_i(\mathbf{y}) dS_y - \sum_{j=N_u+1}^N u_j \int_{\Gamma_q} q^*(\mathbf{x}_k, \mathbf{y}) \Phi_j(\mathbf{y}) dS_y =$$

$$\int_{\Gamma_u} q^*(\mathbf{x}_k, \mathbf{y}) [\bar{u}(\mathbf{y}) - \bar{u}(\mathbf{x}_k)] dS_y - \int_{\Gamma_q} [u^*(\mathbf{x}_k, \mathbf{y}) \bar{q}(\mathbf{y}) + q^*(\mathbf{x}_k, \mathbf{y}) \bar{u}(\mathbf{x}_k)] dS_y. \quad (21)$$

对于边界节点 $\{\mathbf{x}_m\}_{m=N_u+1}^N \in \Gamma_q$ 时,把(19),(20)式代入(18)式,得:

$$\sum_{i=1}^{N_u} q_i \int_{\Gamma_u} u^*(\mathbf{x}_m, \mathbf{y}) \Phi_i(\mathbf{y}) dS_y + \sum_{j=N_u+1}^N u_j \left\{ \int_{\Gamma_u} \Phi_j(\mathbf{x}_m) q^*(\mathbf{x}_m, \mathbf{y}) dS_y - \int_{\Gamma_q} q^*(\mathbf{x}_m, \mathbf{y}) [\Phi_j(\mathbf{y}) - \Phi_j(\mathbf{x}_m)] dS_y \right\} =$$

$$\int_{\Gamma_u} q^*(\mathbf{x}_m, \mathbf{y}) \bar{u}(\mathbf{y}) dS_y - \int_{\Gamma_q} u^*(\mathbf{x}_m, \mathbf{y}) \bar{q}(\mathbf{y}) dS_y. \quad (22)$$

则(21),(22)式的线性方程组为:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_q \end{bmatrix}, \quad (23)$$

其中 $\mathbf{q} = [q_1, q_2, \dots, q_{N_u}]^T$ 是 N_u 个未知量节点 q_i 的向量形式, $\mathbf{u} = [u_{N_u+1}, u_{N_u+2}, \dots, u_N]^T$ 是 $N - N_u$ 个未知量节点 u_j 的向量形式。

(23)式左边的分块矩阵分别为:

$$\mathbf{A}_{11}[k, i] = \int_{\Gamma_u} u^*(\mathbf{x}_k, \mathbf{y}) \Phi_i(\mathbf{y}) dS_y, \quad (24)$$

$$\mathbf{A}_{12}[k, j] = - \int_{\Gamma_q} q^*(\mathbf{x}_k, \mathbf{y}) \Phi_j(\mathbf{y}) dS_y, \quad (25)$$

$$\mathbf{A}_{21}[m, i] = \int_{\Gamma_u} u^*(\mathbf{x}_m, \mathbf{y}) \Phi_i(\mathbf{y}) dS_y, \quad (26)$$

$$\mathbf{A}_{22}[m, j] = \int_{\Gamma_u} \Phi_j(\mathbf{x}_m) q^*(\mathbf{x}_m, \mathbf{y}) dS_y - \int_{\Gamma_q} q^*(\mathbf{x}_m, \mathbf{y}) [\Phi_j(\mathbf{y}) - \Phi_j(\mathbf{x}_m)] dS_y. \quad (27)$$

(24)~(27)式中的 $k, i = 1, \dots, N_u$ 和 $m, j = N_u + 1, \dots, N$ 。(23)式的等式右边的分量为:

$$f_u[k] = \int_{\Gamma_u} q^*(\mathbf{x}_k, \mathbf{y}) [\bar{u}(\mathbf{y}) - \bar{u}(\mathbf{x}_k)] dS_y - \int_{\Gamma_q} [u^*(\mathbf{x}_k, \mathbf{y}) \bar{q}(\mathbf{y}) + q^*(\mathbf{x}_k, \mathbf{y}) \bar{u}(\mathbf{x}_k)] dS_y, \quad (28)$$

$$f_q[m] = \int_{\Gamma_u} q^*(\mathbf{x}_m, \mathbf{y}) \bar{u}(\mathbf{y}) dS_y - \int_{\Gamma_q} u^*(\mathbf{x}_m, \mathbf{y}) \bar{q}(\mathbf{y}) dS_y. \quad (29)$$

(28), (29)式中的 $k = 1, \dots, N_u, m = N_u + 1, \dots, N$ 。

(25), (26), (28)式是正则积分,可用高斯积分计算。(24), (27), (29)式是由正则积分和弱奇异积分组成的,正则积分用高斯积分计算。(23)式可以计算出未知向量 \mathbf{q} 和 \mathbf{u} ,再把计算出来的向量 \mathbf{q} 和 \mathbf{u} 分别代入(19), (20)式,从而可以计算出 Γ_u 上的 q 和 Γ_q 上的 u 。然后把计算出来的 q 和 u 代入(15)式,于是就可以计算出内点 $\mathbf{x} \in \Omega$ 的位势和位势梯度。

3 弱奇异积分的处理和计算

在(24)式中,由于基本解 $u^*(\mathbf{x}_k, \mathbf{y})$ 包含 $\ln[r(\mathbf{x}_k, \mathbf{y})]$,当 $\mathbf{x}_k \in \Gamma_u$ 和 $\mathbf{y} \in \Gamma_u$ 重合时,基本解 $u^*(\mathbf{x}_k, \mathbf{y})$ 会产生弱奇异。在(29)式中,由于基本解 $u^*(\mathbf{x}_m, \mathbf{y})$ 包含 $\ln[r(\mathbf{x}_m, \mathbf{y})]$,当 $\mathbf{x}_m \in \Gamma_q$ 和 $\mathbf{y} \in \Gamma_q$ 重合时,基本解 $u^*(\mathbf{x}_m, \mathbf{y})$ 会产生弱奇异。

对包含 $\ln(r)$ 的弱奇异积分,可采用如下对数积分^[17]:

$$I = \int_{-1}^1 \ln[r(\eta)] f(\eta) d\eta, \quad (30)$$

其中

$$r(\eta) = +\sqrt{s_{11} [x(\eta) - x_p]^2 + 2s_{12} [x(\eta) - x_p] [y(\eta) - y_p] + s_{22} [y(\eta) - y_p]^2}. \quad (31)$$

上式中 $x = \sum_{k=1}^3 x_k \psi_k(\eta), y = \sum_{k=1}^3 y_k \psi_k(\eta)$ 。其中 $(\mathbf{x}_p, \mathbf{y}_p)$ 为源点, $(x(\eta), y(\eta))$ 为场点, $\psi_1(\eta) = \frac{1}{2} \eta(\eta-1), \psi_2(\eta) = 1 - \eta^2, \psi_3(\eta) = \frac{1}{2} \eta(\eta+1)$ 。由于有 $dr \approx ds \approx \mathbf{J} d\eta$,其中 \mathbf{J} 是雅克比矩阵,得:

$$\mathbf{J} = + \left[\left(\frac{dx}{d\eta} \right)^2 + \left(\frac{dy}{d\eta} \right)^2 \right]^{\frac{1}{2}}. \quad (32)$$

其中, $\eta \rightarrow \eta_p, r(\eta) \rightarrow \mathbf{J}_p |\eta - \eta_p|, \mathbf{J}_p = \mathbf{J}(\eta_p)$ 。

I 可表达为:

$$I = \int_{-1}^1 \ln[r(\eta)] [f(\eta) - f(\eta_p)] d\eta + f(\eta_p) \int_{-1}^1 \ln[r(\eta)] d\eta \equiv I_1 + f(\eta_p) I_2, \quad (33)$$

I_2 可表达为:

$$I_2 = \int_{-1}^{\eta_p} \ln[r(\eta)] d\eta + \int_{\eta_p}^1 \ln[r(\eta)] d\eta = \int_{-1}^{\eta_p} \ln \left[\frac{r(\eta)}{\mathbf{J}_p (\eta_p - \eta)} \right] d\eta + \int_{\eta_p}^1 \ln \left[\frac{r(\eta)}{\mathbf{J}_p (\eta - \eta_p)} \right] d\eta = \int_{-1}^{\eta_p} \ln[\mathbf{J}_p (\eta_p - \eta)] d\eta + \int_{\eta_p}^1 \ln[\mathbf{J}_p (\eta - \eta_p)] d\eta. \quad (34)$$

把(34)式中的最后两个积分计算出之后,再将 I_2 代入(33)式中,得:

$$I = \int_{-1}^1 \ln[r(\eta)] [f(\eta) - f(\eta_p)] d\eta + f(\eta_p) \int_{-1}^1 \ln \left[\frac{r(\eta)}{\mathbf{J}_p |\eta - \eta_p|} \right] d\eta +$$

$$f(\eta_p)[2\ln(\mathbf{J}_p) + (1 + \eta_p)[\ln(1 + \eta_p) - 1] + (1 - \eta_p)[\ln(1 - \eta_p) - 1]]. \quad (35)$$

(35)式中的前两个积分是正则积分,可用高斯积分计算。把 $\mathbf{J} = \frac{l}{2}, r(\eta) = \left(\frac{l}{2}\right) |\eta - \eta_p|$ 代入(35)式中,得:

$$I = \int_{-1}^1 \ln\left(\frac{l}{2} |\eta - \eta_p|\right) [f(\eta) - f(\eta_p)] d\eta + f(\eta_p) \left[2\ln\left(\frac{l}{2}\right) + (1 + \eta_p)[\ln(1 + \eta_p) - 1] + (1 - \eta_p)[\ln(1 - \eta_p) - 1] \right]. \quad (36)$$

4 数值算例

考虑一个各向异性长方形区域问题和一个各向异性正方形区域问题的数值算例,来验证本文方法的有效性。

为了验证收敛性,定义误差 $\varepsilon(v) = \sqrt{\frac{\sum_{k=1}^M (v_{\text{num}}^k - v_{\text{exact}}^k)^2}{\sum_{k=1}^M (v_{\text{exact}}^k)^2}}$,其中 M 是计算点的个数, v_{num}^k 和 v_{exact}^k 分别为数值解和解析解。

4.1 各向异性长方形区域问题

各向异性的系数为 $k_{11} = 2, k_{12} = 1, k_{22} = 3$,长方形的上边为 Neumann 条件,其他边为 Dirichlet 条件。该问题的解析解为 $u(x, y) = x^2 - y^2 + xy$ 。当节点数目 $N = 512$ 时(Dirichlet 边界上 384 个节点, Neumann 边界上 128 个节点),长方形的上边边界上的点可以计算出 u ,数值结果见图 1。长方形的其它边边界上的点可以计算出 $\frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$,数值结果见图 2。结果表明,数值解和解析解是吻合的。在进行收敛性分析时,分别采用了 16, 32, 64,

128, 256 和 512 个节点。本文方法和边界元法中的 $u, \frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$ 的误差如图 3。结果表明,误差随着节点个数的增加而降低,本文方法比边界元法的计算精度更高和收敛速度更快。

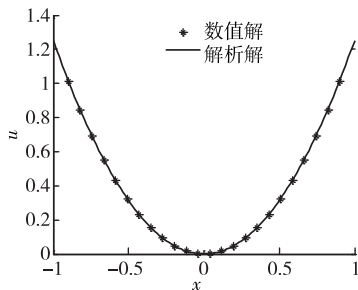


图 1 u 的解析解和数值解

Fig. 1 Numerical and analytical solutions of u

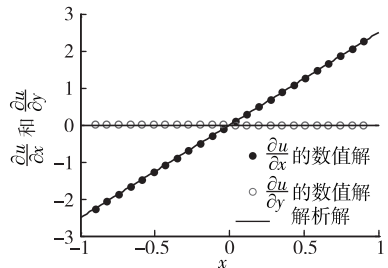


图 2 $\frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$ 的解析解和数值解

Fig. 2 Numerical and analytical solutions of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

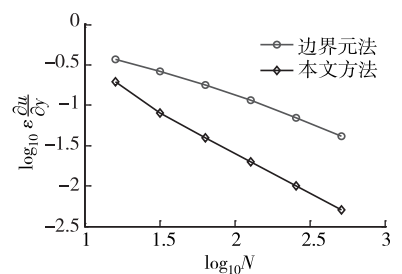
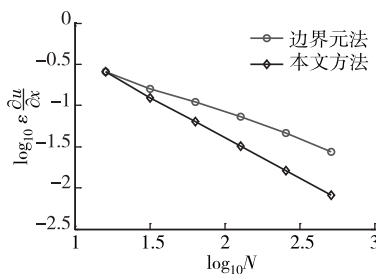
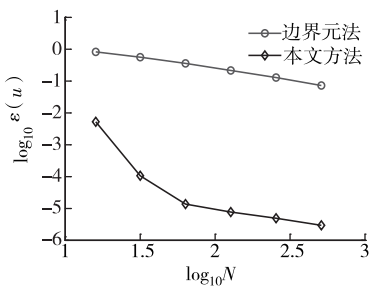


图 3 $u, \frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$ 的收敛性分析

Fig. 3 Convergence analysis of $u, \frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$

4.2 各向异性正方形区域问题

各向异性的系数为 $k_{11} = 1, k_{12} = 2, k_{22} = 5$,正方形的上边为 Neumann 条件,其他边为 Dirichlet 条件。该问

题的解析解为 $u(x, y) = 3x^2 + xy - y^2 + 4x + 5y + 2$ 。当节点数目 $N = 512$ 时 (Dirichlet 边界上 384 个节点, Neumann 边界上 128 个节点), 正方形的上边边界上的点可以计算出 u , 数值结果见图 4。正方形的其它边边界上的点可以计算出 $\frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$, 数值结果见图 5。结果表明, 数值解和解析解是吻合的。在进行收敛性分析时, 分别采用了 16, 32, 64, 128, 256 和 512 个节点。本文方法和边界元法中的 $u, \frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$ 的误差如图 6。结果表明, 误差随着节点个数的增加而降低, 本文方法比边界元法的计算精度更高和收敛速度更快。

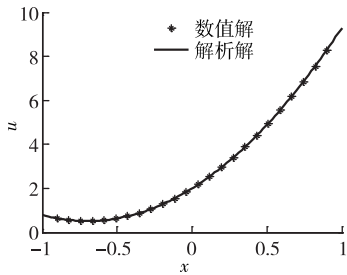


图 4 u 的解析解和数值解

Fig. 4 Numerical and analytical solutions of u

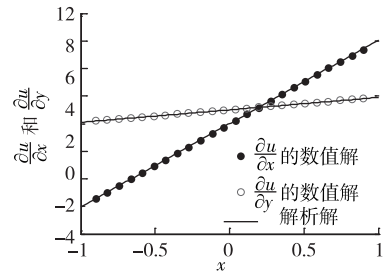


图 5 $\frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$ 的解析解和数值解

Fig. 5 Numerical and analytical u solutions of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

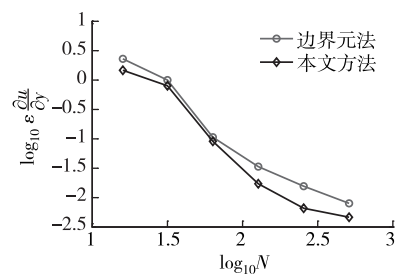
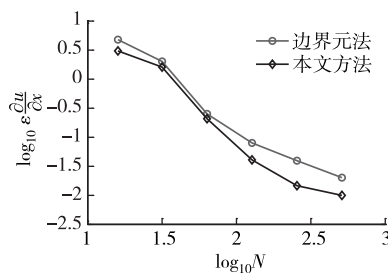
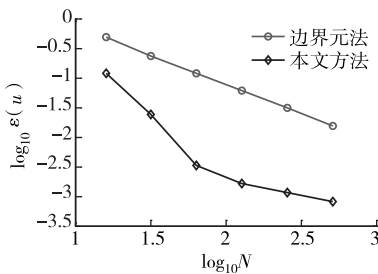


图 6 $u, \frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$ 的收敛性分析

Fig. 3 Convergence analysis of $u, \frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$

5 总结

本文把非奇异权函数的改进移动最小二乘插值法和边界积分方程方法相结合, 提出了二维各向异性位势问题的改进插值型边界无单元法。这种改进插值型边界无单元法是直接采用节点变量的真实解作为基本未知量的。由于非奇异权函数的改进移动最小二乘插值法的形函数有满足 Kronecker δ 函数的性质的特点, 因此能直接施加边界条件。数值算例表明, 本文提出的改进插值型边界无单元法比边界元法的计算精度更高和收敛速度更快。

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An Improved Interpolating Boundary Element-Free Method with Nonsingular Weight Function for Two-Dimensional Anisotropic Potential Problems

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Abstract: [Purposes]Combining the boundary integral equation method with the improved interpolating moving least-square method (IIMLS) based on nonsingular weight function, an improved interpolating boundary element-free method(IIBEFM) is developed for solving two-dimensional anisotropic potential problems. [Methods]On the basis of the improved moving least-square interpolation method, the improved interpolating moving least-square interpolation method with nonsingular weight function is discussed. Its shape function satisfies the property of Kronecker delta function, so boundary conditions can be applied directly. [Findings]Numerical examples show that the method is effective and feasible for solving two-dimensional anisotropic potential problems. [Conclusions] Compared with the boundary element method, this method has better accuracy and convergence.

Keywords: potential problems; improved interpolating moving least-square method with nonsingular weight function; improved interpolating boundary element-free method