四阶抛物型方程的样条子域精细积分配置法*

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摘 要:对于四阶抛物模型方程周期初值问题,可用有限差分方法进行求解。通常的有限差分方法在使用过程中受到精度和稳定性的限制。本文首先将四阶抛物型方程转化为一个二阶的偏微分方程组,然后对时间项采用子域精细积分的方法、空间项采用三次样条基本公式进行离散,得到了一个含参数 $\alpha > 0(\alpha \ll h)$ 的无条件稳定的差分格式,所得到的差分方程为五点、两层隐格式,它的局部截断误差为 $O(\tau^2 + \alpha\tau^2 + h^4)$。$\tau, h$ 分别为时间及空间步长,最后的数值实验表明,本文的方法具有很好的数值精度和良好的实用性。

关键词:四阶抛物型方程;子域精细积分;配置法
中图分类号:O241.8 文献标识码:A 文章编号:1672-6693(2007)04-0033-04

近年来对高阶抛物型方程和方程组的研究逐渐增多。文献[1]中构造了两个格式,一个格式的截断误差为 $O(\tau^2 + h^2)$,另一个为 $O(\tau^2 + h^4)$;文献[2]构造了一族三层(特殊情况下为两层)、含双参数、绝对稳定、高精度、五对角线型的隐式差分格式,其局部截断误差为 $O(\tau^2 + h^6)$;文献[3-4]提出了一类含有多个参数的差分格式,其局部截断误差阶能达到 $O(\tau^2 + h^6)$,但是这些差分格式都比较复杂,且为三层格式,不利于编程和实际问题的计算。

本文考虑四阶抛物型方程周期初边值问题

$$\begin{cases}\dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} = 0, -\infty < x < \infty, 0 \le t \le T\\ u(x+L,t) = u(x,t), -\infty < x < \infty, 0 \le t \le T\\ u(x,0) = f(x), -\infty < x < \infty\end{cases} \tag{1}$$

首先将方程(1)式转化为二阶的偏微分方程组,然后利用钟万勰提出的子域精细积分思想[5-6]和三次样条函数逼近,构造了一个无条件稳定的差分格式,其局部截断误差为 $O(\tau^2 + \alpha\tau^2 + h^4)$,该格式都是两层的隐式格式,其系数矩阵为严格对角占优的五对角矩阵,可用平方根法求解。数值实验表明:本文所提出的格式是有效的,理论分析与实际计算相吻合。

1 子域精细积分配置法

将方程(1)中的四阶抛物型方程改写成等价的二阶偏微分方程组

$$\begin{cases}\dfrac{\partial u}{\partial t} = -\dfrac{\partial^2 v}{\partial x^2} & (2)\\ v = \dfrac{\partial^2 u}{\partial x^2}, 0 < x < 1, 0 < t \le T & (3)\end{cases}$$

取时间步长 τ,空间步长 $h = L/M$(M 为正整数),函数 $u(x,t)$ 在节点 $(x_i, t_n) = (ih, n\tau)$ 处的数值解记为 u_i^n,函数 $v(x,t)$ 在节点 $(x_i, t_n) = (ih, n\tau)$ 处的数值解记为 v_i^n。令 $p = -\dfrac{\partial^2 v}{\partial x^2}$,为了构造精度高而且稳定性好的子域精细积分格式,引入一个附加项 αu_i($\alpha > 0, \alpha \ll h$ 是参数),加到(2)式的两端,得

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + \alpha u_i = q \tag{4}$$

其中 $q = p + \alpha u_i$。如果 q 取某一个常数,则利用常数变易法可解得(4)式的通解

$$u_i(t) = C\mathrm{e}^{-\alpha t} + \frac{q}{\alpha} \tag{5}$$

其中 C 是任意常数。由条件 $u_i(t_n) = u_i^n$,确定该常数 C,并将 $t = t_{n+1}$ 代入(5)式,整理可得

$$u_i^{n+1} = bu_i^n + \frac{1-b}{\alpha}q, b = \mathrm{e}^{-\alpha t} \in (0,1) \tag{6}$$

下面引进三次样条函数逼近,取

$$q = -M_i^{n+1} + \alpha u_i^{n+1} \tag{7}$$

将 q 代入(6)式,整理得

* 收稿日期:2007-04-18 修回日期:2007-06-27
资助项目:广西民族大学研究生教育创新计划(No. gxun-chx0754)
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$$u_i^{n+1} = u_i^n + \frac{1-b}{\alpha b}(-M_i^{n+1}), b = e^{-\alpha t} \in (0,1) \quad (8)$$

其中 M_i^{n+1} 表示函数 $v(x,t)$ 对 x 的二阶偏导数在点 (x_i, t_{n+1}) 的近似值。对(3)式取

$$v_i^{n+1} = N_i^{n+1} \quad (9)$$

其中 N_i^{n+1} 表示函数 $u(x,t)$ 对 x 的二阶偏导数在点 (x_i, t_{n+1}) 的近似值。结合三次样条函数的基本关系式[7],由(8)式和(9)式可推出如下的差分格式

$$\begin{aligned}&(1+36\gamma)u_{i-2}^{n+1} + (8-144\gamma)u_{i-1}^{n+1} + \\ &(18+216\gamma)u_i^{n+1} + (8-144\gamma)u_{i+1}^{n+1} + \\ &(1+36\gamma)u_{i+2}^{n+1} = u_{i-2}^n + 8u_{i-1}^n + 18u_i^n + 8u_{i+1}^n + u_{i+2}^n\end{aligned} \quad (10)$$

其中 $\gamma = (1-b)/\alpha b h^4 > 0, i = 0,1,\cdots,M-1$。

根据周期边界条件上式可写成

$$AU = f \quad (11)$$

其中

$$A = \begin{bmatrix} 18+216\gamma & 8-144\gamma & 1+36\gamma & & & & & 1+36\gamma & 8-144\gamma \\ 8-144\gamma & 18+216\gamma & 8-144\gamma & 1+36\gamma & & & & & 1+36\gamma \\ 1+36\gamma & 8-144\gamma & 18+216\gamma & 8-144\gamma & 1+36\gamma & & & & \\ & 1+36\gamma & 8-144\gamma & 18+216\gamma & 8-144\gamma & 1+36\gamma & & & \\ \square & & & & & & & & \square \\ 1+36\gamma & & & & 1+36\gamma & 8-144\gamma & 18+216\gamma & 8-144\gamma \\ 8-144\gamma & 1+36\gamma & & & & 1+36\gamma & 8-144\gamma & 18+216\gamma \end{bmatrix}$$

$$B = \begin{bmatrix} 18 & 8 & 1 & & & & 1 & 8 \\ 8 & 18 & 8 & 1 & & & & 1 \\ 1 & 8 & 18 & 8 & 1 & & & \\ & 1 & 8 & 18 & 8 & 1 & & \\ \square & & & & & & & \square \\ 1 & & & & 1 & 8 & 18 & 8 \\ 8 & 1 & & & & 1 & 8 & 18 \end{bmatrix}$$

$U = (u_0^{n+1}, u_1^{n+1}, \cdots, u_{M-1}^{n+1})^T, e = (f_0^n, f_1^n, \cdots, f_{M-1}^n)^T$, $f_i^n = u_i^n, f = Be, i = 0,1,\cdots,M-1$。

显然,由差分格式(10)式的系数可看出,求解差分方程(10)式等价于求解严格对角占优的五对角线性方程组(11)式,可用平方根法[8]来求解该方程组。

用 Fourier 方法对(10)式进行稳定性分析,令 $u_j^n = \lambda^n e^{ij\theta}$,将其代入(10)式计算整理得增长因子

$$\lambda = \frac{4(\cos\theta+2)^2}{4(\cos\theta+2)^2 + 144\gamma(1-\cos\theta)^2} \quad (12)$$

显然 $|\lambda| \leqslant 1$ 恒成立,所以格式(10)对任意参数 $\alpha > 0(\alpha \ll h)$ 是无条件稳定的。此外,利用 Taylor 展开式容易求得格式(10)在结点 (x_i, t_n) 处的局部截断误差阶为 $O(\tau^2 + \alpha\tau^2 + h^4)$。

2 数值例子

考虑如下四阶抛物型方程周期初值问题

$$\begin{cases}\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0, (-\infty < x < +\infty, 0 \leqslant t \leqslant T) \\ u(x+2\pi, t) = u(x,t), (-\infty < x < +\infty, 0 \leqslant t \leqslant T) \\ u(x,0) = \sin(x), -\infty < x < +\infty\end{cases} \quad (13)$$

其精确解为

$$u(x,t) = e^{-t}\sin x \quad (14)$$

取 $h = \pi/32, r = 1/2, 2, \tau = rh^4$,利用本文格式(10)计算到 $n = 500$,并与同类方程的差分格式进行比较,其结果如下表1和表2。对表1格式(6)相应时刻的精确解和数值解也进行了比较,如图1、图2、图3所示,图像显示数值解与精确解吻合得很好。

表1 本文格式(10)式的误差结果($n = 500$)

格式	r	x: $5\pi/32$	$22\pi/32$	$39\pi/32$	$56\pi/32$
格式(10)	1/2	1.007E-5	3.007E-5	2.281E-5	2.104E-5
	1	1.682E-5	5.851E-5	4.459E-5	3.650E-5
	2	2.663E-5	1.081E-4	8.359E-5	5.959E-5
Crank-Nicholson 格式	1/2	1.023E-5	3.046E-5	2.311E-5	2.134E-5
	1	1.733E-5	6.012E-5	4.582E-5	3.757E-5
	2	2.828E-5	1.145E-4	8.856E-5	6.326E-5

表 2　文[1] 格式和文[4] 格式的误差结果(n = 500)

格式	r	x			
		5π/32	22π/32	39π/32	56π/32
文献[1] 格式	1/2	1.717E - 5	3.029E - 5	2.311E - 5	2.578E - 5
	2	6.406E - 5	1.130E - 4	8.621E - 5	9.610E - 5
文献[4] 格式	1/2	1.240E - 12	2.203E - 12	1.676E - 12	1.867E - 12
	2	1.030E - 10	1.816E - 10	1.387E - 10	1.546E - 10

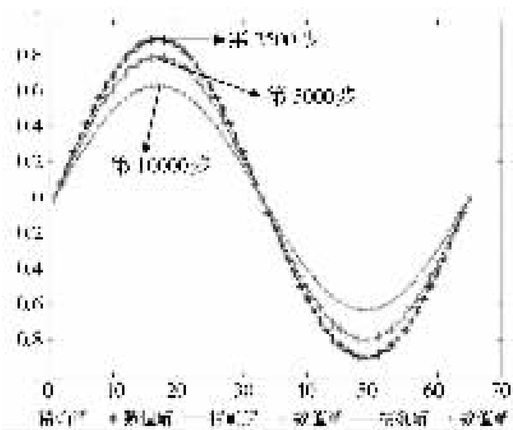

图 1　数值解和精确解在不同时间步的比较
(h = 32/π, r = 1/2)

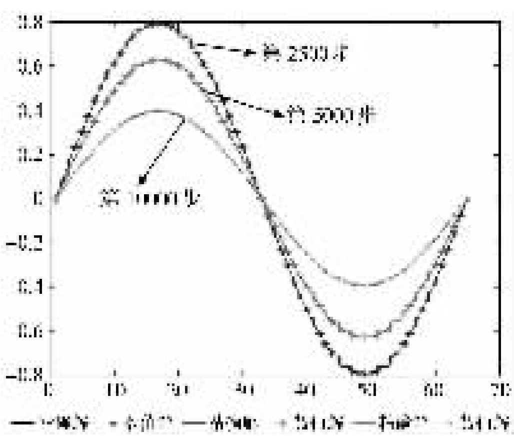

图 2　数值解和精确解在不同时间步的比较
(h = 32/π, r = 1)

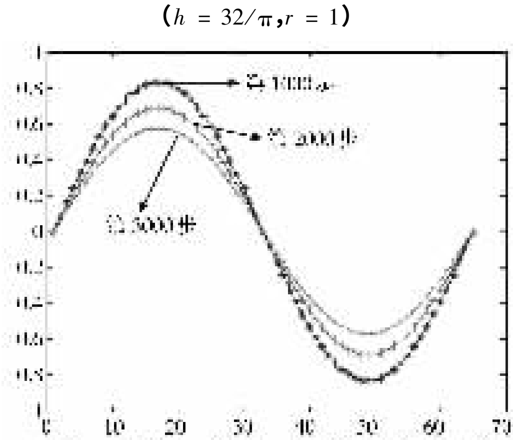

图 3　数值解和精确解在不同时间步的比较
(h = 32/π, r = 2)

3 结果与讨论

1) 文献[1] 格式的局部截断误差阶最高为 $O(\tau^2 + h^2)$,而本文格式的局部截断误差阶为 $O(\tau^2 + \alpha\tau^2 + h^4)$,显然,从表 1 和表 2 的结果看出,本文的格式比文[1] 的格式精度要高,与经典的 Crank-Nicholson 格式相比,精度也有所提高,这与理论相吻合。

2) 文献[4] 格式的精度比较高,但是其绝对稳定的条件受到了限制,且格式为含有多个参数的三层格式,在计算实际问题时,必须用其他方法计算出第二层上的函数值,不利于实际问题的计算,且只有当参数满足某些条件时,才能保证无条件稳定。从图 1、图 2、图 3 的结果可以看出,本文的格式同样具有很高的精度,且格式简单,为两层的隐格式,其系数矩阵为严格对角占优五对角矩阵,可用平方根法进行求解。可见,本文的格式更具有实用价值。

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Sub-domain Precise Integration Spline Collocation Method for Solving Four Order Parabolic Equation

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Abstract: At present, some researchers have got a lot of good numerical solutions for the periodic initial value problem of four order parabolic equation: such as finite difference method, finite elements method, spectral Galerkin method and so on. Of the methods, the finite difference method is used mostly. However, the method is restricted with numerical stability and precision in the course of using it. However, sometimes the method is not conducive to solve the practical problem well. So given unconditional a stable and high precision method is of great significance. And many experts and scholars are researching it all the time. In this paper based on sub-domain precise integration method, an unconditional stable sub-domain precise integration implicit scheme containing parameter $\alpha > 0 (\alpha \ll h)$ is presented. Rubin, a foreign expert, put forward the cubic spline collocation for numerical solutions of partial differential equation first. And some people have been researching collocation for numerical solutions partial differential equation from that time on. They have given a 3×3 matrix system which can be got solution directly. They generalize the alternating direction implicit of difference method to spline function. At same time they research high precise computation scheme. Being flowers, some foreign and civil experts have further developed the spline collocation method. They have got some good results. But cubic spline collocation being used in periodic initial value problem in four order parabolic equation has not example so far. Therefore, this paper uses sub-domain precise integration idea and cubic spline collocation method to solve initial value problem of four order parabolic equation. It brings forth a new idea. The paper is arranged as follows in detail: First, the parabolic equation is transferred to second order partial differential equation set; next, the equation is here discrete by using sub-domain precise integration method for time and using cubic spline basic formula for space. And the difference equation can be solved by the method of forward elimination and backward substitution. The stability condition of the spline sub-domain precise integration method system is discussed, the local truncation error is $O(\tau^2 + \alpha\tau^2 + h^4)$, where τ and h represent the time step and space step respectively. The method does not need iterative method matrix computing. The total in number or amount of calculation is small. To solve the problem in this way is thought conveniently. The result shows the collocation method in this paper is better than the method of Saul´ev, and it´s precision is higher than Crank-Nicholson classical scheme. This scheme in this paper is efficiency and practicable. The numerical example at the end of this paper is given. It has shown that the numerical result of practical computing accords with the theoretic analysis. The method is of much practical value.

Key words: four order parabolic equation; sub-domain precise integration; collocation method

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