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具有对数势能的粘性 Cahn-Hilliard 方程的有限元算法*

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摘要:【目的】研究具有对数势的粘性 Cahn-Hilliard 方程的有限元数值求解方法。【方法】通过正则化处理，将对数势函数 $F(u)$ 的定义域从 $(-1,1)$ 扩展到 $(-\infty,+\infty)$；提出了一种新的数值格式，即在时间上采用二阶格式，在空间上采用混合有限元方法进行离散；通过理论分析，证明了该数值格式是无条件能量稳定的，并对它进行了误差估计分析。【结果】给出了几个数值算例，对理论部分进行了验证。【结论】实验结果表明理论分析与数值算例的结果一致。

关键词:对数势能；粘性 Cahn-Hilliard 方程；混合有限元方法；二阶格式

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Cahn-Hilliard 方程最初由 Cahn 和 Hilliard 在研究热力学中两种物质之间的扩散现象时提出[1-3]，它是一类非常重要的四阶非线性扩散方程。随着研究的不断深入，Cahn-Hilliard 方程已应用于许多领域中，相应地也有许多关于求解该方程数值方法的研究。文献[4]提出了线性有限差分格式，证明了该格式的唯一可解性，但没有研究它能量衰减的性质；文献[5]提出了 Crank-Nicolson 格式，该格式基于时间半离散，研究了能量稳定性和收敛性；文献[6]通过隐式向后微分格式导出二阶时间精度，并证明该格式是无条件能量稳定以及收敛的；文献[7]讨论了具有对数势的 Allen-Cahn 和 Cahn-Hilliard 方程的误差估计。粘性 Cahn-Hilliard 方程[8-9]建立在 Cahn-Hilliard 方程的基础上的，它最早被用来描述二元合金、玻璃等混合物冷却过程中相变的动力学行为。本文提出了粘性 Cahn-Hilliard 方程的二阶全离散格式，在时间离散上采用了 Crank-Nicolson 格式，空间上运用了混合有限元方法，验证了所提格式是无条件能量稳定的，并给出了严格的误差估计。

本文提出了所研究方程的模型及变量，基于混合有限元方法给出了具有二阶格式的粘性 Cahn-Hilliard 方程的半离散格式和全离散格式，并证明了所提出格式的能量稳定性，进行了误差估计分析，最后给出了数值算例。

1 模型及变量

具有对数势的粘性 Cahn-Hilliard 方程为:

$$\begin{cases} u_t=\Delta w,(x,t)\in\Omega\times(0,T] \\ w=f(u)-\varepsilon^2\Delta u+\beta u_t,(x,t)\in\Omega\times(0,T] \\ \partial_{\boldsymbol{n}}u=\partial_{\boldsymbol{n}}w=0,(x,t)\in\partial\Omega\times(0,T] \\ u(\cdot,0)=u_0,x\in\Omega \end{cases},\tag{1}$$

其中:u 表示混合物中某种物质的浓度，w 是化学势，$\Omega\in\mathbf{R}^d$，$d=2$，$\varepsilon>0$ 是界面参数，$\boldsymbol{n}$ 是边界的单位外法向量。$\beta>0$ 是粘性系数，当 $\beta=0$ 时，方程(1)就是经典的 Cahn-Hilliard 方程。f 是对数势函数 F 的导数[10]，即 $\forall\theta\in(0,1)$，有:

$$F(u)=\frac{\theta}{2}((1+u)\ln(1+u)+(1-u)\ln(1-u))+\frac{1}{2}(1-u^2),$$

$$f(u)=F'(u)=\frac{\theta}{2}(\ln(1+u)-\ln(1-u))-u。$$

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根据正则化思想,本文考虑下面的对数势函数[11]:$\forall \kappa\in(0,1)$,

$$\breve{F}(u):=\begin{cases}\frac{\theta}{2}(1+u)\ln(1+u)+\frac{\theta}{4\kappa}(1-u^2)+\frac{\theta}{2}(1-u)\ln\kappa-\frac{\theta\kappa}{4}+\frac{1}{2}(1-u^2), u\geqslant 1-\kappa\\ \frac{\theta}{2}((1+u))\ln(1+u)+(1-u)\ln(1-u))+\frac{1}{2}(1-u^2), |u|<1-\kappa\\ \frac{\theta}{2}(1-u)\ln(1-u)+\frac{\theta}{4\kappa}(1+u^2)+\frac{\theta}{2}(1+u)\ln\kappa-\frac{\theta\kappa}{4}+\frac{1}{2}(1-u^2), u\leqslant -1+\kappa\end{cases},$$

$\breve{F}(u)$的导数为:

$$\breve{f}:=\breve{F}'(u)=\begin{cases}\frac{\theta}{2}\ln(1+u)+\frac{\theta}{2}-\frac{\theta}{2\kappa}(1-u)-\frac{\theta}{2}\ln\kappa-u, u\geqslant 1-\kappa\\ \frac{\theta}{2}(\ln(1+u)-\ln(1-u))-u, |u|<1-\kappa\\ -\frac{\theta}{2}\ln(1-u)-\frac{\theta}{2}+\frac{\theta}{2\kappa}(1+u)+\frac{\theta}{2}\ln\kappa-u, u\leqslant -1+\kappa\end{cases},$$

下文将用正则化的对数势函数$\breve{F}(u)$及它的导数$\breve{f}$来代替 F 和 f,并且为了简单起见,仍记作 F 和 f。Cahn-Hilliard 的自由能函数为:$E=\int_\Omega\left(\frac{\varepsilon^2}{2}|\nabla u|^2+F(u)\right)\mathrm{d}x$。Cahn-Hilliard 方程是质量守恒的,即$(u(\cdot,t),1)=(u_0,1)$,且能量是耗散的,$\frac{\mathrm{d}E}{\mathrm{d}t}=-\int_\Omega|\nabla w|^2\mathrm{d}x\leqslant 0$。

2 离散格式

$L^2(\Omega)$是平方可积函数空间,内积 $(u,v)=\int_\Omega u(x)v(x)\mathrm{d}x$,范数$\|u\|=\|u\|_{L^2}=\sqrt{(u,u)}$。$H^1(\Omega)$是 Sobolev 空间,半范 $|u|_{H^1}=\left(\int_\Omega|Du|^2\mathrm{d}x\right)^{\frac{1}{2}}$,范数 $\|u\|_{H^1(\Omega)}=\left(\int_\Omega|u|^2\mathrm{d}x+\int_\Omega|Du|^2\mathrm{d}x\right)^{\frac{1}{2}}$。

2.1 半离散格式

具有对数势的粘性 Cahn-Hilliard 方程的弱解形式为:

$$(u_t,v)+(\nabla w,\nabla v)=0,\ \forall v\in H^1(\Omega),\tag{2}$$

$$(w,v)-(f(u),v)-\varepsilon^2(\nabla u,\nabla v)-\beta(u_t,v)=0,\ \forall v\in H^1(\Omega)。\tag{3}$$

将时间区间$[0,T]$进行剖分,$0=t_0<t_1<\cdots<t_N=T$,T 是一个正整数,时间节点满足 $t_i=i\tau, i=0,1,\cdots,N$,$\tau=\frac{T}{N}$为时间步长。构造具有对数势的粘性 Cahn-Hilliard 方程的半离散格式[12],即给定 u^n,u^{n-1},求 $u^{n+1},w^{n+\frac{1}{2}}$,当 $n\geqslant 1$ 时,满足:$(\delta_\tau u^{n+1},v)+(\nabla w^{n+\frac{1}{2}},\nabla v)=0$,$(w^{n+\frac{1}{2}},v)-(f(\tilde{u}^{n+\frac{1}{2}}),v)-\varepsilon^2(\nabla\hat{u}^{n+\frac{1}{2}},\nabla v)-\beta(\delta_\tau u^{n+1},v)=0$。其中:$\delta_\tau u^{n+1}=\frac{u^{n+1}-u^n}{\tau}$,$\tilde{u}^{n+\frac{1}{2}}=\frac{3}{2}u^n-\frac{1}{2}u^{n-1}$,$\hat{u}^{n+\frac{1}{2}}=\frac{3}{4}u^{n+1}+\frac{1}{4}u^{n-1}$。

2.2 全离散格式

设 $T_h=K$ 是区域Ω 上的拟一致剖分,h_i 表示网格大小,$h=\max\limits_{0\leqslant i\leqslant N}h_i$,$S_h$ 是分片连续的有限元空间,定义为 $S_h=\{v_h\in C(\Omega)\,|\,|v_h|_K\in P_k(x,y),K\in T_h\}\subset H^1(\Omega)$,其中,$P_k(x,y)$是 x,y 的次数不超过 $k\in\mathbf{Z}^+$ 的多项式的集合。定义 $L_0^2:=\{u\in L^2(\Omega)\,|\,(u,1)=0\}$,$\hat{S}_h:=S_h\cap L_0^2$。构造具有对数势的粘性 Cahn-Hilliard 方程的全离散格式,即给定 u_h^n,u_h^{n-1},求 $u_h^{n+1},w_h^{n+\frac{1}{2}}$,当 $n\geqslant 1$ 时,满足:

$$(\delta_\tau u_h^{n+1},v_h)+(\nabla w_h^{n+\frac{1}{2}},\nabla v_h)=0,\tag{4}$$

$$(w_h^{n+\frac{1}{2}},v_h)-(f(\tilde{u}_h^{n+\frac{1}{2}}),v_h)-\varepsilon^2(\nabla\hat{u}_h^{n+\frac{1}{2}},\nabla v_h)-\beta(\delta_\tau u_h^{n+1},v_h)=0。\tag{5}$$

其中:$\delta_\tau u_h^{n+1}=\frac{u_h^{n+1}-u_h^n}{\tau}$,$\tilde{u}_h^{n+\frac{1}{2}}=\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}$,$\hat{u}_h^{n+1}=\frac{3}{4}u_h^{n+1}+\frac{1}{4}u_h^{n-1}$。

为了格式的能量稳定性[5],在(5)式中加入辅助项 $A\tau\Delta(u_h^{n+1}-u_h^n)$和 $B(u_h^{n+1}-2u_h^n+u_h^{n-1})$,这里 $A\geqslant\frac{L^2}{4}$,$B\geqslant$

$\frac{L}{2}$，得：

$$(w_h^{n+1},v_h)-\left(f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right),v_h\right)-\varepsilon^2\left(\nabla\left(\frac{3}{4}u_h^{n+1}+\frac{1}{4}u_h^{n-1}\right),v_h\right)-\beta(\delta_\tau u_h^{n+1},v_h)-$$
$$A\tau(\nabla(u_h^{n+1}-u_h^n),\nabla v_h)-B(u_h^{n+1}-2u_h^n+u_h^{n-1},v_h)=0 \tag{6}$$

Ritz 算子 $R_h:H^1(\Omega)\to S_h$ 满足：$\begin{cases}(\nabla(u-R_hu),\nabla v)=0\\(R_hu-u,1)=0\end{cases}$。

3 稳定性分析

定义 1　离散的 Laplacian 算子 $\Delta_h,S_h\to\hat{S}_h$ 是可逆的，对 $\forall\, v_h\in S_h,\Delta_h v_h\in\hat{S}_h$，有：

$$(\nabla(-\Delta_h^{-1})v_h,\nabla\chi)=(v_h,\chi),\ \forall\,\chi\in S_h。$$

定义 2[13]　Ritz 投影算子满足估计：$\|u-R_h(u)\|+h\|u-R_h(u)\|_{H^1(\Omega)}\leqslant Ch^{q+1}\|u\|_{H^{q+1}(\Omega)}$。

定义 3　H^{-1} 范数为：$\|v_h\|_{-1,h}:=\sqrt{(v_h,(-\Delta_h)^{-1}v_h)}=\sup\limits_{0\neq\chi\in\hat{S}_h}\frac{(v,\chi)}{\|\nabla\chi\|},\ \forall\,v_h\in\hat{S}_h$。

引理 1[14]　假设 $\chi,\psi\in\hat{S}_h(\Omega)$，有：

$$(\chi,\psi)_{-1,h}:=(\nabla T_h(\chi),\nabla T_h(\psi))=(\chi,T_h(\psi))=(T_h(\chi),\psi)。$$

其中：$T_h:\hat{S}_h(\Omega)\to\hat{S}_h(\Omega)$是可逆线性算子，$(\nabla T_h(\chi),\nabla\nu)=(\chi,\nu)$，且满足：$|(\chi,g)|\leqslant\|\chi\|_{-1,h}\|\nabla g\|$。其中：$\chi\in\hat{S}_h,g\in S_h(\Omega)$。

基于已有文献中有关能量稳定性的研究思想[6]，进行如下能量估计。

定理 1　令$(u_h^{n+1},w_h^{n+\frac{1}{2}})$是(4)式和(6)式的解，定义：

$$\Xi(u_h^{n+1},u_h^n):=E(u_h^{n+1})+\frac{\varepsilon^2}{8}\|\nabla u_h^{n+1}-\nabla u_h^n\|^2+\left(\frac{B}{2}+\frac{L}{4}\right)\|u_h^{n+1}-u_h^n\|^2,$$

则对任意的 $\tau,h,\varepsilon>0$，当 $n\geqslant1$，数值格式(4)式和(6)式满足：

$$\Xi(u_h^{n+1},u_h^n)+\frac{3\tau}{4}\|\nabla w_h^{n+\frac{1}{2}}\|^2+\left(\sqrt{A}-\frac{L}{2}+\frac{\beta}{\tau}\right)\|u_h^{n+1}-u_h^n\|^2+\frac{\varepsilon^2}{8}\|\nabla u_h^{n+1}-2\,\nabla u_h^n+\nabla u_h^{n-1}\|^2+$$
$$\left(\frac{B}{2}-\frac{L}{4}\right)\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2\leqslant\Xi(u_h^n,u_h^{n-1})。 \tag{7}$$

其中：$A\geqslant\frac{L^2}{4},B\geqslant\frac{L}{2}$。

证明　在(4)式中，令 $v_h=\tau w_h^{n+\frac{1}{2}}$，得：

$$(\delta_\tau u_h^{n+1},\tau w_h^{n+\frac{1}{2}})+\tau\|\nabla w_h^{n+\frac{1}{2}}\|^2=0。 \tag{8}$$

在(6)式中，令 $v_h=-(u_h^{n+1}-u_h^n)$，得：

$$-(w_h^{n+\frac{1}{2}},u_h^{n+1}-u_h^n)+\left(f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right),u_h^{n+1}-u_h^n\right)+\varepsilon^2\left(\nabla\left(\frac{3}{4}u_h^{n+1}+\frac{1}{4}u_h^{n-1}\right),\nabla(u_h^{n+1}-u_h^n)\right)+$$
$$\frac{\beta}{\tau}\|u_h^{n+1}-u_h^n\|^2+A\tau\|\nabla(u_h^{n+1}-u_h^n)\|^2+\frac{B}{2}(\|u_h^{n+1}-u_h^n\|^2-\|u_h^n-u_h^{n-1}\|^2)+\frac{B}{2}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2=0。 \tag{9}$$

然后将(8)式和(9)式相加，得：

$$\tau\|\nabla w_h^{n+\frac{1}{2}}\|^2+\left(f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right),u_h^{n+1}-u_h^n\right)+\varepsilon^2\left(\nabla\left(\frac{3}{4}u_h^{n+1}+\frac{1}{4}u_h^{n-1}\right),\nabla(u_h^{n+1}-u_h^n)\right)+$$
$$\frac{\beta}{\tau}\|u_h^{n+1}-u_h^n\|^2+A\tau\|\nabla(u_h^{n+1}-u_h^n)\|^2+\frac{B}{2}(\|u_h^{n+1}-u_h^n\|^2-\|u_h^n-u_h^{n-1}\|^2)+\frac{B}{2}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2=0。 \tag{10}$$

对于 $f(u)$，根据泰勒展开，有：

$$F(u_h^{n+1})=F(\tilde{u}_h^{n+\frac{1}{2}})+f(\tilde{u}_h^{n+\frac{1}{2}})(u_h^{n+1}-\tilde{u}_h^{n+\frac{1}{2}})+\frac{1}{2}f'(\eta_1)(u_h^{n+1}-\tilde{u}_h^{n+\frac{1}{2}})^2, \tag{11}$$

$$F(u_h^n)=F(\tilde{u}_h^{n+\frac{1}{2}})+f(\tilde{u}_h^{n+\frac{1}{2}})(u_h^n-\tilde{u}_h^{n+\frac{1}{2}})+\frac{1}{2}f'(\eta_2)(u_h^n-\tilde{u}_h^{n+\frac{1}{2}})^2。 \tag{12}$$

其中:$\eta_1\in[\tilde{u}_h^{n+\frac{1}{2}},u_h^{n+1}]$,$\eta_2\in[u_h^n,\tilde{u}_h^{n+\frac{1}{2}}]$。记 $f'(\eta_1)=L_1$,$f'(\eta_2)=L_2$,且记 $L=\max\{L_1,L_2\}$。

将(11)式减去(12)式,得:

$$F(u_h^{n+1})-F(u_h^n)=f(\tilde{u}_h^{n+\frac{1}{2}})(u_h^{n+1}-u_h^n)+\frac{1}{2}f'(\eta_1)(u_h^{n+1}-\tilde{u}_h^{n+\frac{1}{2}})^2-\frac{1}{2}f'(\eta_2)(u_h^n-\tilde{u}_h^{n+\frac{1}{2}})^2=$$
$$f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right)(u_h^{n+1}-u_h^n)+\frac{1}{2}f'(\eta_1)(u_h^{n+1}-u_h^n)(u_h^{n+1}-2u_h^n-u_h^{n-1})-$$
$$\frac{1}{8}(f'(\eta_2)-f'(\eta_1))(u_h^n-u_h^{n-1})^2=f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right)(u_h^{n+1}-u_h^n)-\frac{1}{4}f'(\eta_1)\|u_h^n-u_h^{n-1}\|^2+$$
$$\frac{1}{4}f'(\eta_1)\|u_h^{n+1}-u_h^n\|^2+\frac{1}{4}f'(\eta_1)\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2-\frac{1}{8}(f'(\eta_2)-f'(\eta_1))\|u_h^n-u_h^{n-1}\|^2=$$
$$f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right)(u_h^{n+1}-u_h^n)-\frac{1}{8}(f'(\eta_1)+f'(\eta_2))\|u_h^n-u_h^{n-1}\|^2+$$
$$\frac{1}{4}f'(\eta_1)\|u_h^{n+1}-u_h^n\|^2+\frac{1}{4}f'(\eta_1)\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2。$$

因此,

$$f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right)(u_h^{n+1}-u_h^n)\geqslant(F(u_h^{n+1})-F(u_h^n),1)-\frac{L}{4}\|u_h^n-u_h^{n-1}\|^2-$$
$$\frac{L}{4}\|u_h^{n+1}-u_h^n\|^2-\frac{L}{4}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2\geqslant$$
$$(F(u_h^{n+1})-F(u_h^n),1)+\frac{L}{4}(\|u_h^{n+1}-u_h^n\|^2-\|u_h^n-u_h^{n-1}\|^2)-\frac{L}{4}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2-\frac{L}{2}\|u_h^{n+1}-u_h^n\|^2。\tag{13}$$

又因为:

$$\left(\nabla\left(\frac{3}{4}u_h^{n+1}+\frac{1}{4}u_h^{n-1}\right),\nabla(u_h^{n+1}-u_h^n)\right)=$$
$$\frac{1}{4}(\nabla(u_h^{n+1}-2u_h^n+u_h^{n-1})(\nabla(u_h^{n+1}-u_h^n))+\frac{1}{2}(\|\nabla u_h^{n+1}\|^2-\|\nabla u_h^n\|^2)=$$
$$\frac{1}{8}(\|\nabla u_h^{n+1}-\nabla u_h^n\|^2-\|\nabla u_h^n-\nabla u_h^{n-1}\|^2)+\frac{1}{8}(\|\nabla u_h^{n+1}-2\,\nabla u_h^n+\nabla u_h^{n-1}\|)+\frac{1}{2}(\|\nabla u_h^{n+1}\|^2-\|\nabla u_h^n\|^2)。\tag{14}$$

在(4)式中令 $v_h=\sqrt{A}\tau(u_h^{n+1}-u_h^n)$,根据柯西不等式得:

$$\sqrt{A}\tau\|u_h^{n+1}-u_h^n\|^2=-\sqrt{A}\tau(\nabla w_h^{n+\frac{1}{2}},\nabla(u_h^{n+1}-u_h^n))\leqslant\frac{\tau}{4}\|\nabla w_h^{n+\frac{1}{2}}\|^2+A\tau\|\nabla(u_h^{n+1}-u_h^n)\|^2。\tag{15}$$

将(13)~(15)式代入(10)式得到:

$$\frac{3\tau}{4}\|w_h^{n+\frac{1}{2}}\|^2+\frac{L}{4}(\|u_h^{n+1}-u_h^n\|^2-\|u_h^n-u_h^{n-1}\|^2)-\frac{L}{4}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2-\frac{L}{2}\|u_h^{n+1}-u_h^n\|^2+$$
$$(F(u_h^{n+1})-F(u_h^n),1)+\frac{\varepsilon^2}{8}\|\nabla u_h^{n+1}-2\,\nabla u_h^n+\nabla u_h^{n-1}\|^2+\frac{\varepsilon^2}{2}(\|\nabla u_h^{n+1}\|^2-\|\nabla u_h^n\|^2)+$$
$$\frac{\varepsilon^2}{8}(\|\nabla u_h^{n+1}-\nabla u_h^n\|^2-\|\nabla u_h^n-\nabla u_h^{n-1}\|^2)+\frac{\beta}{\tau}\|u_h^{n+1}-u_h^n\|^2+\sqrt{A}\tau\|u_h^{n+1}-u_h^n\|^2+$$
$$\frac{B}{2}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2+\frac{B}{2}(\|u_h^{n+1}-u_h^n\|^2-\|u_h^n-u_h^{n-1}\|^2)\leqslant 0。$$

则定理1得证。 证毕

定理2 设 $E(u_h^0)\leqslant C_0$,存在常数 $C>0$,对任意的 $\tau,h>0$,则有估计:

$$\begin{cases}\sum_{i=1}^{n}\tau\|w_h^{i+\frac{1}{2}}\|^2\leqslant C\\ \|\nabla u_h^{n+1}\|^2\leqslant C\\ \sum_{i=1}^{n}\|u_h^{i+1}-u_h^i\|^2\leqslant C\\ \sum_{i=1}^{n}\|u_h^{i+1}-2u_h^i+u_h^{i-1}\|^2\leqslant C\end{cases}。\tag{16}$$

证明　将(7)式从 1 到 n 求和，则(16)式可得证。　　证毕

注　定理 1 中的 A 和 B 只是在理论上辅助证明了格式的能量稳定性，但在数值案例中，当 τ 取很小时，A,B 可以取到 0，即：在数值模拟时 A,B 取值的误差估计对结果影响不大。

4 误差估计

为了简单起见，引入以下符号：

$$\widetilde{\xi}_u^{n+1}=u^{n+1}-R_h u^{n+1},\hat{\xi}_u^{n+1}=R_h u^{n+1}-u_h^{n+1},\sigma(u^{n+1})=u_t^{n+1}-\delta_\tau R_h u^{n+1},$$
$$\widetilde{\xi}_w^{n+\frac{1}{2}}=w^{n+\frac{1}{2}}-R_h w^{n+\frac{1}{2}},\hat{\xi}_w^{n+\frac{1}{2}}=R_h w^{n+\frac{1}{2}}-w_h^{n+\frac{1}{2}}。$$

对于(u,w)做如下正则性假设：

$$u\in L^\infty(0,T;W^{1,5}(\Omega))\cap L^\infty(0,T;H^{q+1}(\Omega)),u_{ttt}\in L^\infty(0,T;L^2(\Omega)),w\in L^\infty(0,T;H^{q+1}(\Omega))。$$

引理 2[15]　对于充分光滑的函数 $u(t)=u(\cdot,t)\in C^3[0,T]$，在 t_{n+1} 时，$\delta_\tau u^{n+1}=\dfrac{u^{n+1}-u^n}{\tau}$ 是二阶收敛的，则：$u_t^{n+1}=\delta_\tau u^{n+1}+R_t^{n+1}$，$\|R_t^{n+1}\|\leqslant C\tau^2\max\limits_{t\in[0,T]}\|u_{ttt}\|$，其中 C 与 τ 无关。

引理 3　假设(u,w)是(2)，(3)式的解，则有估计：$\|\sigma(u^{n+1})\|^2\leqslant Ch^{2q+2}+C\tau^4$。

证明　由引理 2 以及定义 2 可得：

$$\|\sigma(u^{n+1})\|^2=\|u_t^{n+1}-\delta_\tau u^{n+1}+\delta_\tau u^{n+1}-\delta_\tau R_h u^{n+1}\|^2=\|R_t^{n+1}+\delta_\tau u^{n+1}-\delta_\tau R_h u^{n+1}\|^2\leqslant$$
$$2\|R_t^{n+1}\|^2+2\|\delta_\tau u^{n+1}-\delta_\tau R_h u^{n+1}\|^2\leqslant C\tau^4+h^{2q+2}\|\delta_\tau u^{n+1}\|_{H^{q+1}}^2。$$

由积分中值定理可得：

$$\|\delta_\tau u^{n+1}\|_{H^{q+1}}^2=\left\|\frac{1}{\tau}(u^{n+1}-u^n)\right\|_{H^{q+1}}^2=\left\|\frac{1}{\tau}\int_{t_n}^{t_{n+1}}u_t\,\mathrm{d}t\right\|_{H^{q+1}}^2\leqslant\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\|u_t\|_{H^{q+1}}\mathrm{d}t\right)^2\leqslant$$
$$\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\|u_t\|_{W^{1,\infty}(0,T,H^{q+1})}\mathrm{d}t\right)^2\leqslant\left(\frac{1}{\tau}C(t_{n+1}-t_n)\right)^2\leqslant\frac{1}{\tau^2}C\tau^2\leqslant C。$$

从而命题得证。　　证毕

基于已有文献中有关格式误差估计的研究思想[6]，可以将该格式(4)式和(6)式进行如下误差估计。

定理 3　初始问题(2)式和(3)式与全离散格式(4)式和(6)式的解分别是(u,w)和$(u_h^{n+1},w_h^{n+\frac{1}{2}})$，则存在常数 C,τ,h，使得：$\sum\limits_{i=1}^{n}C\tau\|\nabla\hat{\xi}_w^{i+\frac{1}{2}}\|^2+\dfrac{3\varepsilon^2}{4}\|\nabla\hat{\xi}_u^{n+1}\|^2+\sum\limits_{i=1}^{n}\beta\tau\|\delta_\tau\hat{\xi}_u^{i+1}\|^2\leqslant C\tau^4+Ch^{2q}$。

证明　在 $t=n+1$ 处，(2)式和(3)式分别减去(4)式和(6)式，可得：

$$(\sigma(u^{n+1}),v)+(\delta_\tau\hat{\xi}_u^{n+1},v)+(\nabla\hat{\xi}_w^{n+\frac{1}{2}},\nabla v)=0,\tag{17}$$

$$(\widetilde{\xi}_w^{n+\frac{1}{2}}+\hat{\xi}_w^{n+\frac{1}{2}},v)-(f(u_h^{n+1}),v)+\left(f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right),v\right)-\frac{3}{4}\varepsilon^2(\nabla\hat{\xi}_u^{n+1},\nabla v)-$$
$$\frac{1}{4}\varepsilon^2(\nabla(u^{n+1}-u_h^{n-1}),\nabla v)-\beta(\sigma(u^{n+1}),v)-\beta(\delta_\tau\hat{\xi}_u^{n+1},v)+A\tau(\nabla(u_h^{n+1}-u_h^n),\nabla v)+B(u_h^{n+1}-2u_h^n+u_h^{n-1},v)=0。\tag{18}$$

在(17)式中令 $v=\hat{\xi}_w^{n+\frac{1}{2}}$，在(18)式中令 $v=-\delta_\tau\hat{\xi}_u^{n+1}$，然后将两式相加，得：

$$\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2+\frac{3\tau^2}{8\tau}(\|\nabla\hat{\xi}_u^{n+1}\|^2-\|\nabla\hat{\xi}_u^n\|^2+\|\nabla\hat{\xi}_u^{n+1}-\nabla\hat{\xi}_u^n\|^2)+\beta\|\delta_\tau\hat{\xi}_u^{n+1}\|^2=$$
$$-(\sigma(u^{n+1}),\hat{\xi}_w^{n+\frac{1}{2}})+(\widetilde{\xi}_w^{n+\frac{1}{2}},\delta_\tau\hat{\xi}_u^{n+1})-(f(u^{n+1}),\delta_\tau\hat{\xi}_u^{n+1})+\left(f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right),\delta_\tau\hat{\xi}_u^{n+1}\right)-$$
$$\frac{\varepsilon^2}{4}(\nabla(u^{n+1}-u_h^{n-1}),\nabla\delta_\tau\hat{\xi}_u^{n+1})-\beta(\sigma(u^{n+1}),\delta_\tau\hat{\xi}_u^{n+1})-A\tau(\nabla(u_h^{n+1}-u_h^n),\nabla\delta_\tau\hat{\xi}_u^{n+1})+$$
$$B(u_h^{n+1}-2u_h^n+u_h^{n-1},\delta_\tau\hat{\xi}_u^{n+1})=\sum_{i=1}^{7}M_i。$$

其中：

$$M_1=-(\sigma(u^{n+1}),\hat{\xi}_w^{n+\frac{1}{2}}),\ M_2=(\hat{\xi}_w^{n+\frac{1}{2}},\delta_\tau\hat{\xi}_u^{n+1}),\ M_3=-(f(u^{n+1}),\delta_\tau\hat{\xi}_u^{n+1})+\left(f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right),\delta_\tau\hat{\xi}_u^{n+1}\right),$$

$$M_4=-\frac{\varepsilon^2}{4}(\nabla(u^{n+1}-u_h^{n-1}),\nabla\delta_\tau\hat{\xi}_u^{n+1}),M_5=-\beta(\sigma(u^{n+1}),\delta_\tau\hat{\xi}_u^{n+1}),$$

$$M_6=-A\tau(\nabla(u_h^{n+1}-u_h^n),\nabla\delta_\tau\hat{\xi}_u^{n+1}),M_7=B(u_h^{n+1}-2u_h^n+u_h^{n-1},\delta_\tau\hat{\xi}_u^{n+1})。$$

接下来估计 M_i,根据 Poincare 不等式、Cauchy-Schwarz 不等式、Young 不等式和引理 3,可得:

$$M_1\leqslant|(\sigma(u^{n+1}),\hat{\xi}_w^{n+\frac{1}{2}})|\leqslant\|\sigma(u^{n+1})\|\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|\leqslant C\|\sigma(u^{n+1})\|^2+\frac{1}{2}\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2\leqslant C\tau^4+Ch^{2q+2}+\frac{1}{2}\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2。$$

根据定义 2、引理 1 和 Young 不等式可得:

$$M_2\leqslant|(\tilde{\xi}_w^{n+\frac{1}{2}},\delta_\tau\hat{\xi}_u^{n+1})|\leqslant\|\nabla\tilde{\xi}_w^{n+\frac{1}{2}}\|\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}\leqslant\frac{1}{\alpha}\|\nabla\tilde{\xi}_w^{n+\frac{1}{2}}\|^2+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2\leqslant Ch^{2q}+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2。$$

相似地,对于 M_3 有估计:

$$M_3\leqslant\left|\left(f(u^{n+1})-f\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1}\right),\delta_\tau\hat{\xi}_u^{n+1}\right)\right|\leqslant\left|\left(f'(\eta)\left(u^{n+1}-\frac{3}{2}u_h^n+\frac{1}{2}u_h^{n-1}\right),\delta_\tau\hat{\xi}_u^{n+1}\right)\right|\leqslant$$

$$\left\|\nabla\left(u^{n+1}-\frac{3}{2}u_h^n+\frac{1}{2}u_h^{n-1}\right)\right\|\|f'(\eta)\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}\leqslant\frac{L^2}{\alpha}\left\|\nabla\left(u^{n+1}-\frac{3}{2}u_h^n+\frac{1}{2}u_h^{n-1}\right)\right\|^2+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2\leqslant$$

$$\frac{L^2}{\alpha}\left\|\nabla\left(u^{n+1}-\frac{3}{2}u^n+\frac{3}{2}u^n-\frac{3}{2}u_h^n+\frac{1}{2}u^{n-1}-\frac{1}{2}u^{n-1}+\frac{1}{2}u_h^{n-1}\right)\right\|^2+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2\leqslant$$

$$C\tau^4+Ch^{2q}+C\|\nabla\hat{\xi}_u^n\|^2+C\|\nabla\hat{\xi}_u^{n-1}\|^2+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2。$$

根据 Cauchy-Schwarz 不等式、Young 不等式和引理 1 可得:

$$M_4\leqslant\frac{\varepsilon^2}{4}|(\nabla(u^{n+1}-u_h^{n+1}),\nabla\delta_\tau\hat{\xi}_u^{n+1})|\leqslant$$

$$\frac{\varepsilon^2}{4}|(\nabla(u^{n+1}-2u^n+u^{n-1}),\nabla\delta_\tau\hat{\xi}_u^{n+1})|+\frac{\varepsilon^2}{4}|(\nabla(u^{n-1}-u_h^{n-1}),\nabla\delta_\tau\hat{\xi}_u^{n+1})|\leqslant$$

$$\frac{\varepsilon^2}{\alpha}\|\nabla\Delta(u^{n+1}-2u^n+u^{n-1})\|^2+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2+\frac{1}{8}\|\nabla(u^{n-1}-u_h^{n-1})\|^2+\frac{\varepsilon^2}{8}\|\nabla\delta_\tau\hat{\xi}_u^{n+1}\|^2\leqslant$$

$$C\tau^4+Ch^{2q}+C\|\nabla\hat{\xi}_u^{n-1}\|^2+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2+\frac{\varepsilon^2}{8}\|\nabla\delta_\tau\hat{\xi}_u^{n+1}\|^2。$$

根据 Cauchy-Schwarz 不等式、Young 不等式和引理 3 可得:

$$M_5\leqslant|\beta(\sigma^{n+1},\delta_\tau\hat{\xi}_u^{n+1})|\leqslant\beta\|\sigma^{n+1}\|\|\delta_\tau\hat{\xi}_u^{n+1}\|\leqslant\frac{\beta}{2}\|\sigma^{n+1}\|^2+\frac{\beta}{2}\|\delta_\tau\hat{\xi}_u^{n+1}\|^2\leqslant C\tau^4+Ch^{2q+2}+\frac{\beta}{2}\|\delta_\tau\hat{\xi}_u^{n+1}\|^2。$$

根据引理 1 和 Young 不等式可得:

$$M_6=A\tau(\nabla(u_h^{n+1}-u_h^n),\nabla\delta_\tau\hat{\xi}_u^{n+1})=A\tau(\nabla(u_h^{n+1}-u^{n+1}+u^{n+1}-u^n+u^n-u_h^n),\nabla\delta_\tau\hat{\xi}_u^{n+1})\leqslant$$

$$A\|\nabla\hat{\xi}_u^{n+1}-\nabla\hat{\xi}_u^n\|+\frac{C\tau^2}{\alpha}\|\nabla\Delta(u^{n+1}-u^n)\|^2+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2\leqslant$$

$$C\tau^4+A\|\nabla\hat{\xi}_u^{n+1}\|^2+C\|\nabla\hat{\xi}_u^n\|^2+\frac{\alpha}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2。$$

类似地,对于 M_7 有:

$$M_7=B|(u_h^{n+1}-2u_h^n+u_h^{n-1},\delta_\tau\hat{\xi}_u^{n+1})|\leqslant\|\nabla(u_h^{n+1}-2u_h^n+u_h^{n-1})\|\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}\leqslant$$

$$B\|\nabla(u_h^{n+1}-2u_h^n+u_h^{n-1})\|^2+\frac{\alpha B^2}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2\leqslant C\|\nabla(u^{n+1}-2u^n+u^{n-1})\|^2+$$

$$\frac{4}{\alpha}\|\nabla(u^{n+1}-u_h^{n+1})\|^2+C\|\nabla(u^n-u_h^n)\|^2+C\|\nabla(u^{n-1}-u_h^{n-1})\|^2+\frac{\alpha B^2}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2\leqslant$$

$$C\tau^4+Ch^{2q}+\frac{4}{\alpha}\|\nabla\hat{\xi}_u^{n+1}\|^2+C\|\nabla\hat{\xi}_u^n\|^2+C\|\nabla\hat{\xi}_u^{n-1}\|^2+\frac{\alpha B^2}{4}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2。$$

综合上述不等式可得:

$$\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2+\frac{3\varepsilon^2}{8\tau}(\|\nabla\hat{\xi}_u^{n+1}\|^2-\|\nabla\hat{\xi}_u^n\|^2+\|\nabla\hat{\xi}_u^{n+1}-\nabla\hat{\xi}_u^n\|^2)+\beta\|\delta_\tau\nabla\hat{\xi}_u^{n+1}\|^2\leqslant$$

$$C\tau^4+Ch^{2q}+\left(A+\frac{8}{\alpha}\right)\|\nabla\hat{\xi}_u^{n+1}\|^2\leqslant$$

$$C\|\nabla\hat{\xi}_u^n\|^2+\|\nabla\hat{\xi}_u^{n-1}\|^2+\frac{1}{2}\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2+\frac{\beta}{2}\|\delta_\tau\nabla\hat{\xi}_u^{n+1}\|^2+\left(\alpha+\frac{\alpha B^2}{4}\right)\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2+\frac{\varepsilon^2}{8}\|\delta_\tau\nabla\hat{\xi}_u^{n+1}\|^2。\tag{19}$$

接下来估计$\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2$。由引理 1，在(17)式中令 $v=T_h(\delta_\tau\hat{\xi}_u^{n+1})$，并运用不等式：

$$\|T_h(\delta_\tau\hat{\xi}_u^{n+1})\|^2\leqslant C\|\nabla T_h(\delta_\tau\hat{\xi}_u^{n+1})\|^2=C\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2$$

可得：

$$(\delta_\tau\hat{\xi}_u^{n+1},T_h(\delta_\tau\hat{\xi}_u^{n+1}))=\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2\leqslant|(\sigma(u^{n+1}),T_h(\delta_\tau\hat{\xi}_u^{n+1}))|+|(\nabla\hat{\xi}_w^{n+\frac{1}{2}},T_h(\delta_\tau\hat{\xi}_u^{n+1}))|\leqslant$$
$$\|\sigma(u^{n+1})\|^2+\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2+\frac{1}{2}\|T_h(\delta_\tau\hat{\xi}_u^{n+1})\|^2\leqslant C\tau^4+Ch^{2q+2}+\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2+\frac{1}{2}\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2。$$

因此，有：

$$\|\delta_\tau\hat{\xi}_u^{n+1}\|_{-1,h}^2\leqslant C\tau^4+Ch^{2q+2}+2\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2。\tag{20}$$

选择适当的 $\alpha(0\leqslant\alpha\leqslant\frac{1}{4+B^2})$，将(20)式代入(19)式，并乘以 2τ，整理得：

$$2\tau\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2+\frac{3\varepsilon^2}{4}(\|\nabla\hat{\xi}_u^{n+1}\|^2-\|\nabla\hat{\xi}_u^n\|^2+\|\nabla\hat{\xi}_u^{n+1}-\nabla\hat{\xi}_u^n\|^2)+\beta\tau\|\delta_\tau\hat{\xi}_u^{n+1}\|^2\leqslant$$
$$C\tau\tau^4+C\tau h^{2q}+2\left(A+\frac{8}{\alpha}\right)\tau\|\nabla\hat{\xi}_u^{n+1}\|^2+C\tau\|\nabla\hat{\xi}_u^n\|^2+C\tau\|\nabla\hat{\xi}_u^{n-1}\|^2+(1+4\alpha+\alpha B^2)\tau\|\nabla\hat{\xi}_w^{n+\frac{1}{2}}\|^2+\frac{\varepsilon^2}{4}\|\nabla\delta_\tau\hat{\xi}_u^{n+1}\|^2。$$

上式从 1 到 n 求和，并取 $0\leqslant\tau\leqslant\min\left\{\sqrt{3},\frac{3\alpha\varepsilon^2}{8\alpha A+64}\right\}$，根据离散的 Gronwall 不等式，得：

$$\sum_{i=1}^n\tau\|\nabla\hat{\xi}_w^{i+\frac{1}{2}}\|^2+\frac{3\varepsilon^2}{4}\|\nabla\hat{\xi}_u^{n+1}\|^2+\sum_{i=1}^n\beta\tau\|\delta_\tau\hat{\xi}_u^{i+1}\|^2\leqslant C\tau^4+Ch^{2q}。$$

从而定理 3 得证。　　证毕

5 数值分析

对于粘性 Cahn-Hilliard 方程，本文得到的误差估计在时间和空间上均为二阶精度，而大多已有文献中的数值分析都是研究了相对误差的空间收敛阶，没有对时间收敛阶进行分析。本文得到的空间和时间的收敛阶均为二阶，相对于一阶收敛阶的精度更高。

在数值实验部分，采用了一些数值算例验证理论分析的正确性及有效性。选择初始条件为：

$$u_0=0.24\cos(2\pi x)\cos(2\pi y)+0.4\cos(\pi x)\cos(3\pi y),$$

计算区域为(−1，1)×(−1，1)。

5.1 空间收敛阶

在表 1～4 中，选择固定的参数 $\tau=0.01$，变化的网格步长分别为 $h=\frac{1}{8},\frac{1}{16},\frac{1}{32},\frac{1}{64}$，$\kappa=0.01$，相对误差 $\|\hat{\xi}_u\|_{H^1}$ 的空间收敛阶接近于 2，与理论分析得到的收敛阶保持一致，且不同的 θ 和 β 对收敛阶的影响不大。

表 1　$\varepsilon=0.1,\theta=0.1,\beta=0$ 时 $\hat{\xi}_u$ 的空间收敛阶

Tab. 1　The spacial convergence rate of $\hat{\xi}_u$ with $\varepsilon=0.1,\theta=0.1,\beta=0$

h	$A=0,B=0$		$A=100,B=100$	
	$\|\hat{\xi}_u\|_{H^1}$	收敛阶	$\|\hat{\xi}_u\|_{H^1}$	收敛阶
$\frac{1}{8}$	0.751 986		0.202 064	
$\frac{1}{16}$	0.193 726	2.181 52	0.043 464 3	2.216 91
$\frac{1}{32}$	0.042 705 6	2.116 98	0.010 085 4	2.107 57
$\frac{1}{64}$	0.009 844 86	2.069 03	0.002 414 78	2.062 3

表 2　$\varepsilon=0.1,\theta=0.1,\beta=0.1$ 时 $\hat{\xi}_u$ 的空间收敛阶

Tab. 2　The spacial convergence rate of $\hat{\xi}_u$ with $\varepsilon=0.1,\theta=0.1,\beta=0.1$

h	$A=0,B=0$		$A=100,B=100$	
	$\|\hat{\xi}_u\|_{H^1}$	收敛阶	$\|\hat{\xi}_u\|_{H^1}$	收敛阶
$\frac{1}{8}$	0.839 911		0.957 334	
$\frac{1}{16}$	0.194 187	2.112 79	0.209 957	2.188 93
$\frac{1}{32}$	0.043 572 9	2.155 94	0.045 226 9	2.214 84
$\frac{1}{64}$	0.010 092 2	2.110 2	0.010 488 6	2.108 35

表 3 $\varepsilon=0.1,\theta=0.7,\beta=0$ **时 $\hat{\xi}_u$ 的空间收敛阶**

Tab. 3 The spacial convergence rate of $\hat{\xi}_u$ with $\varepsilon=0.1,\theta=0.7,\beta=0$

h	$A=0,B=0$		$A=100,B=100$	
	$\Vert\hat{\xi}_u\Vert_{H^1}$	收敛阶	$\Vert\hat{\xi}_u\Vert_{H^1}$	收敛阶
$\frac{1}{8}$	0.473 83		0.954 897	
$\frac{1}{16}$	0.113 505	2.061 61	0.209 275	2.189 94
$\frac{1}{32}$	0.024 767	2.196 27	0.045 072	2.215 1
$\frac{1}{64}$	0.005 811 3	2.091 49	0.010 453 3	2.108 27

表 4 $\varepsilon=0.1,\theta=0.7,\beta=0.1$ **时 $\hat{\xi}_u$ 的空间收敛阶**

Tab. 4 The spacial convergence rate of $\hat{\xi}_u$ with $\varepsilon=0.1,\theta=0.7,\beta=0.1$

h	$A=0,B=0$		$A=100,B=100$	
	$\Vert\hat{\xi}_u\Vert_{H^1}$	收敛阶	$\Vert\hat{\xi}_u\Vert_{H^1}$	收敛阶
$\frac{1}{8}$	0.794 323		0.955 039	
$\frac{1}{16}$	0.179 824	2.143 14	0.209 308	2.189 93
$\frac{1}{32}$	0.040 240 1	2.159 88	0.045 079 4	2.215 09
$\frac{1}{64}$	0.009 370 21	2.102 48	0.010 455	2.108 27

5.2 时间收敛阶

在表 5 中,选择参数 $\varepsilon=0.1,\theta=0.12,\beta=0.9,h=\tau=\frac{1}{8},\frac{1}{16},\frac{1}{32},T=1.0$。相对误差 $\Vert\hat{\xi}_u\Vert_{H^1}$ 的时间收敛阶接近于 2,与理论分析得到的收敛阶保持一致。

表 5 $\varepsilon=0.1,\theta=0.12,\beta=0.9$ **时 $\hat{\xi}_u$ 的时间收敛阶**

Tab. 5 The time convergence rate of $\hat{\xi}_u$ with $\varepsilon=0.1,\theta=0.12,\beta=0.9$

τ	$A=0,B=0$		$A=50,B=50$		$A=40,B=70$		$A=70,B=40$	
	$\Vert\hat{\xi}_u\Vert_{H^1}$	收敛阶	$\Vert\hat{\xi}_u\Vert_{H^1}$	收敛阶	$\Vert\hat{\xi}_u\Vert_{H^1}$	收敛阶	$\Vert\hat{\xi}_u\Vert_{H^1}$	收敛阶
$\frac{1}{8}$	0.745 046		0.958 974		0.958 838		0.959 133	
$\frac{1}{16}$	0.187 892	1.987 43	0.210 014	2.191 01	0.209 983	2.191 01	0.210 056	2.190 96
$\frac{1}{32}$	0.041 633 7	2.174 08	0.048 703 2	2.108 4	0.049 956	2.071 55	0.047 304 2	2.150 73

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Finite Element Method for the Viscous Cahn-Hilliard Equation with Logarithmic Potential

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Abstract: [**Purposes**]A finite element method for solving viscous Cahn-Hilliard equations with logarithmic potential is presented. [**Methods**]For the logarithmic potential functions, the domain of the function $F(u)$ extends from $(-1,1)$ to $(-\infty,+\infty)$ by regularization. Then a novel numerical scheme is proposed, and the second order scheme is adopted for the time, and the mixed finite element method is adopted for discretization in the space. Through theoretical analysis, the proposed numerical scheme is proved to be unconditionally energy stable and the error estimates are calculated. [**Findings**]Finally, several numerical examples are given to verify the theoretical parts. [**Conclusions**]The results show that the theoretical analysis is consistent with the results of the numerical examples.

Keywords: logarithmic potential; viscous Cahn-Hilliard equation; mixed finite element method; second order scheme

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