DOI:10.11721/cqnuj20220503

三阶中立型分布时滞微分方程的振动性*

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摘要:【目的】研究 $\beta\geqslant\alpha$ 和 $\beta<\alpha$ 两种情形下一类三阶中立型分布时滞微分方程解的振动性。【方法】利用广义 Riccati 变换技术和 Yang 不等式等方法对该问题进行研究。【结果】建立了该方程每个解振动或收敛到 0 的充分条件。【结论】所得结果推广和改进了已有文献中一些熟知的振动准则，并举例说明了所得结果的适用性。

关键词:振动准则；中立型微分方程；广义 Riccati 变换
中图分类号:O175.26　　**文献标志码**:A　　**文章编号**:1672-6693(2022)06-0092-07

考虑一类三阶中立型分布时滞微分方程：

$$(r(t)|z''(t)|^{\alpha-1}z''(t))'+\int_c^d q(t,\xi)x^{\beta}(\sigma(t,\xi))\mathrm{d}\xi=0,t>t_0。\tag{1}$$

其中：$z(t)=x(t)-\int_a^b p(t,\xi)x(\tau(t,\xi))\mathrm{d}\xi$，$\alpha,\beta$ 是正常数。

本文总假设下列条件成立：

(H_1) $r(t)\in C^1([t_0,\infty),(0,\infty))$，$p(t,\xi)\in C([t_0,\infty)\times[a,b],\mathbf{R})$，$p(t,\xi)\geqslant 0$，$\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(t)\mathrm{d}t=\infty$，$0\leqslant\int_a^b p(t,\xi)\mathrm{d}\xi\leqslant p_0<1$；

(H_2) $\tau(t,\xi)\in C([t_0,\infty)\times[a,b],\mathbf{R})$，$\lim\limits_{t\to\infty}\inf\tau(t,\xi)=\infty$，$\tau(t,\xi)\leqslant t$；

(H_3) $\sigma(t,\xi)\in C^1([t_0,\infty)\times[c,d],\mathbf{R})$，$\lim\limits_{t\to\infty}\inf\sigma(t,\xi)=\infty$，$\sigma(t,\xi)\leqslant t$，$\sigma(t,\xi)$关于 ξ 是增函数；

(H_4) $q(t,\xi)\in C([t_0,\infty]\times[c,d],[0,\infty))$。

微分方程(1)在人口动力学、工程技术等领域都有着广泛应用[1-2]，因此，方程(1)的振动性问题的研究越来越受到关注，取得了一些重要的结果[3-13]。但是，这些结果都是针对 $\alpha=\beta$，$\beta\geqslant\alpha$，$\beta<\alpha$ 这 3 种情况之一给出的，文献[3-11]研究了 $\alpha=\beta$ 的特殊情况，文献[12-13]研究了 $\beta\geqslant\alpha$，$\beta<\alpha$ 中的一种情形。本文的目的是利用广义 Riccati 变换技术，对 $\beta\geqslant\alpha$，$\beta<\alpha$ 这两种情况进行研究，导出微分方程(1)的新的 Riccati 不等式，并将这两种情况统一起来，最终获得方程(1)每个解振动或收敛到 0 的充分条件，所得结果推广和改进了文献[4,7,9,11]的主要结果，并举例说明了本文所得结果的适用性。

1 引理

引理 1　设 $x(t)$是微分方程(1)的最终正解，则可能有以下 4 种情形：

(Ⅰ) $(r(t)|z''(t)|^{\alpha-1}z''(t))'\leqslant 0$，$z''(t)<0$，$z'(t)<0$，$z(t)<0$；

(Ⅱ) $(r(t)|z''(t)|^{\alpha-1}z''(t))'\leqslant 0$，$z''(t)>0$，$z'(t)>0$，$z(t)>0$；

(Ⅲ) $(r(t)|z''(t)|^{\alpha-1}z''(t))'\leqslant 0$，$z''(t)>0$，$z'(t)<0$，$z(t)>0$；

* **收稿日期**:2021-06-19　**修回日期**:2022-05-01　**网络出版时间**:2022-09-19 10:48
资助项目:国家自然科学基金(No. 11361048)；云南省教育厅科学基金(No. 2022J0792；No. 2022J0805)；曲靖市教育科学规划联合项目(No. QJQSKT2021YB02；No. QJQSKT2021ZD02)；曲靖市教育科学规划项目(No. QJ21YB028)
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网络出版地址:https://kns.cnki.net/kcms/detail/50.1165.N.20220916.1817.023.html

(Ⅳ) $(r(t)|z''(t)|^{\alpha-1}z''(t))'\leqslant 0, z''(t)>0, z'(t)<0, z(t)<0$。

证明　由 $x(t)$是方程(1)的最终正解,则 $\exists t_1>t_0$,使得:$x(\sigma(t,\xi))>0, x(\tau(t,\xi))>0, x(t)>0, t>t_1$。由方程(1)有:

$$(r(t)|z''(t)|^{\alpha-1}z''(t))'\leqslant 0, t>t_1。\tag{2}$$

因此,$r(t)|z''(t)|^{\alpha-1}z''(t)$是一个减函数,于是 $z''(t)$定号,即 $\exists t_2>t_1$, $\forall t\geqslant t_2$,有 $z''(t)<0$ 或 $z''(t)>0$。

1) 如果 $z''(t)<0, t\geqslant t_2$,由(2)式可得$(r(t)(-z''(t))^{\alpha})'\geqslant 0, t\geqslant t_2$。则 $r(t)(-z''(t))^{\alpha}$ 是一个增函数,$\exists t_3\geqslant t_2$, $\forall t>t_3$,有 $r(t)(-z''(t))^{\alpha}\geqslant r(t_3)(-z''(t_3))^{\alpha}=K, t>t_3$。即 $z''(t)\leqslant -K^{\frac{1}{\alpha}}r^{-\frac{1}{\alpha}}(t), t>t_3$。

将上式在$[t_3, t]$上积分,得到 $z'(t)\leqslant z'(t_3)-K^{\frac{1}{\alpha}}\int_{t_3}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s$。

令 $t\to\infty$,有$\lim\limits_{t\to\infty}z'(t)=-\infty$,于是 $\exists t_4\geqslant t_3$, $\forall t\geqslant t_4$, $z'(t)<0$;又 $z''(t)<0, t\geqslant t_2$。因此,$\exists t_5\geqslant t_4$, $\forall t\geqslant t_5, z(t)<0$,得到情形(Ⅰ)。

2) 如果 $z''(t)>0, t\geqslant t_2$,于是 $z'(t)$定号,即 $\exists t_6>t_2$, $\forall t\geqslant t_6$,有 $z'(t)<0$ 或者 $z'(t)>0$。

若 $z'(t)<0, t\geqslant t_6$,则 $\exists t_7\geqslant t_6$, $\forall t\geqslant t_7, z(t)>0$ 或者 $z(t)<0$,得到情形(Ⅲ)或情形(Ⅳ)。

若 $z'(t)>0, t\geqslant t_6$,则 $\exists t_8>t_6$, $\forall t\geqslant t_8, z(t)>0$,得到情形(Ⅱ)。　　证毕

2 主要结果

为方便研究,引进两个集合:$E=\{(t,s):t\geqslant s\geqslant t_0\}$, $E_0=\{(t,s):t>s\geqslant t_0\}$。

让 $G(t,s)\in C(E,R)$,则称 $G(t,s)\in X$,如果满足两个条件:1) $G(t,t)=0$, $t\geqslant t_0$;$G(t,s)>0, (t,s)\in E_0$。2) $\dfrac{\partial G(t,s)}{\partial s}\leqslant 0$,且 $\exists\rho\in C^1(I,\mathbf{R}^+), g\in C^1(E_0,\mathbf{R}), I=[t_0,\infty)$,使得:

$$\frac{\rho'(s)}{\rho(s)}G(t,s)+\frac{\partial G(t,s)}{\partial s}=-g(t,s)G^{\frac{\lambda}{1+\lambda}}(t,s), (t,s)\in E_0。$$

定理 1　设 $x(t)$是方程(1)的最终正解,且 $\exists\rho\in C^1(I,\mathbf{R}^+), g\in C^1(E_0,\mathbf{R}), G\in X$,使得:

$$\limsup_{t\to\infty}\frac{1}{G(t,t_0)}\int_{t_0}^{t}\left(G(t,s)\rho(s)\overline{Q}(s)-\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}(A(s))^{\lambda}}\cdot|g(t,s)|^{\lambda+1}\right)\mathrm{d}s=\infty。\tag{3}$$

其中:

$$\overline{Q}(t)=\min\{Q_1(t),Q_2(t)\};Q_i(t)=Q(t)K_i(t), i=1,2;$$

$$Q(t)=\int_c^d q(t,\xi)\mathrm{d}\xi;A(t)=\min\{A_1(t),A_2(t)\};$$

$$A_1(t)=\frac{\alpha}{(r(t)\rho(t))^{\frac{1}{\alpha}}};A_2(t)=\frac{\beta M_2}{(r(t)\rho(t))^{\frac{1}{\beta}}};M_2=(z''(t_8))^{-\frac{\alpha-\beta}{\beta}};$$

$$K_1(t)=M_1\left(\frac{\int_{t_{10}}^{\sigma_1(t)}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\alpha};K_2(t)=\left(\frac{\int_{t_{10}}^{\sigma_1(t)}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\beta};$$

$$M_1=(z(\sigma_1(t_9))^{\beta-\alpha};\sigma_1(t)=\sigma(t,c);\lambda=\min\{\alpha,\beta\}。$$

则引理 1 中的情形(Ⅱ)不存在。

证明　由 $x(t)$是方程(1)的最终正解,则 $\exists t_1>t_0$,使得 $x(\tau(\sigma,\xi))>0, x(\tau(t,\xi))>0, x(t)>0, t>t_1$。

用反证法,设 $z(t)$满足引理 1 的情形(Ⅱ),则由方程(1),当 $t>t_1$ 时,有:

$$(r(t)|z''(t)|^{\alpha-1}z''(t))'=-\int_c^d x^{\beta}(\sigma(t,\xi))q(t,\xi)\mathrm{d}\xi\leqslant-\int_c^d z^{\beta}(\sigma(t,\xi))q(t,\xi)\mathrm{d}\xi\leqslant$$

$$-z^{\beta}(\sigma(t,c))\int_c^d q(t,\xi)\mathrm{d}\xi=-Q(t)z^{\beta}(\sigma_1(t))。$$

其中:$Q(t)=\int_c^d q(t,\xi)\mathrm{d}\xi$, $\sigma_1(t)=\sigma(t,c)$。

下面分别讨论 $\beta\geqslant\alpha$ 和 $\beta<\alpha$ 两种情形。

1) 假设 $\beta\geqslant\alpha$,通过广义 Riccati 变换定义函数 $w(t)=\dfrac{\rho(t)r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}}$。于是 $w(t)>0$,且有:

$$w'(t)=\frac{\rho'(t)r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}}+\rho(t)\frac{(r(t)(z''(t))^{\alpha})'(z'(t))^{\alpha}-\alpha r(t)(z''(t))^{\alpha+1}(z'(t))^{\alpha-1}}{(z'(t))^{2\alpha}}\leqslant$$

$$\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)(z(\sigma_1(t))^{\beta-\alpha}\left(\frac{z(\sigma_1(t)}{z'(t)}\right)^{\alpha}-\frac{\alpha}{(r(t)\rho(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t),t\geqslant t_8。$$

由条件(H_3)知 $\exists t_9\geqslant t_8$,$\forall t\geqslant t_9$,有 $\sigma(t)\geqslant t_8$,则有 $t\geqslant\sigma(t)\geqslant t_8$,$t\geqslant t_9$。

由引理1的情形(Ⅱ):$\forall t\geqslant t_8$,$z(t)>0$,$z'(t)>0$,于是$(z(\sigma_1(t))^{\beta-\alpha}\geqslant(z(\sigma_1(t_9))^{\beta-\alpha}=M_1>0$,$t\geqslant t_9$。其中:$M_1=(z(\sigma_1(t_9))^{\beta-\alpha}$,则有:

$$z'(t)=z'(t_9)+\int_{t_9}^{t}z''(u)\mathrm{d}u=z'(t_9)+\int_{t_9}^{t}\frac{(r(u)(z''(u))^{\alpha})^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(u)}\mathrm{d}u\geqslant r^{\frac{1}{\alpha}}(t)z''(t)\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u,t\geqslant t_8。$$

于是,对 $\forall t\geqslant t_9$ 有:

$$\left(\frac{z'(t)}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)'=\frac{z''(t)\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u-z'(t)r^{-\frac{1}{\alpha}}(t)}{\left(\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\right)^2}\leqslant$$

$$\frac{z''(t)\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u-r^{-\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)z''(t)\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}{\left(\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\right)^2}=0。$$

则 $\dfrac{z'(t)}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}$ 是一个减函数($t\geqslant t_9$),由条件(H_3)可得 $\dfrac{z'(\sigma_1(t))}{\int_{t_9}^{\sigma_1(t)}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\geqslant\dfrac{z'(t)}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}$。即:

$$\frac{z'(\sigma_1(t))}{z'(t)}\geqslant\frac{\int_{t_9}^{\sigma_1(t)}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u},t\geqslant t_9。$$

还可得到:$\exists t_{10}\geqslant t_9$,$\forall t\geqslant t_{10}$,有:

$$z(t)=z(t_{10})+\int_{t_{10}}^{t}z'(u)\mathrm{d}u=$$

$$z(t_{10})+\int_{t_{10}}^{t}\frac{z'(u)}{\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u\geqslant\frac{z'(t)}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\cdot\int_{t_{10}}^{t}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u,t\geqslant t_{10}。$$

则得到 $\dfrac{z(t)}{z'(t)}\geqslant\dfrac{\int_{t_{10}}^{t}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s},t\geqslant t_{10}$。

因此,有:

$$\left(\frac{z(\sigma_1(t))}{z'(t)}\right)^{\alpha}=\left(\frac{z(\sigma_1(t))}{z'(\sigma_1(t))}\cdot\frac{z'(\sigma_1(t))}{z'(t)}\right)^{\alpha}\geqslant\left(\frac{\int_{t_{10}}^{\sigma_1(t)}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u}{\int_{t_9}^{\sigma_1(t)}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\cdot\frac{\int_{t_9}^{\sigma_1(t)}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\alpha}=$$

$$M_1\left(\frac{\int_{t_{10}}^{\sigma_1(t)}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\alpha}\cdot\frac{1}{M_1}=\frac{K_1(t)}{M_1},t\geqslant t_{10}。\tag{4}$$

其中:$K_1(t)=M_1\left(\dfrac{\int_{t_{10}}^{\sigma_1(t)}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\alpha}$。于是有:

$$w'(t)\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-Q(t)K_1(t)\rho(t)-\frac{\alpha}{(r(t)\rho(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t)=$$

$$\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q_1(t)-A_1(t)w^{\frac{\alpha+1}{\alpha}}(t),t\geqslant t_{10}。$$

其中:$Q_1(t)=Q(t)K_1(t),A_1(t)=\dfrac{\alpha}{(r(t)\rho(t))^{\frac{1}{\alpha}}}$。因此有:

$$\rho(t)Q_1(t)\leqslant-w'(t)+\frac{\rho'(t)}{\rho(t)}w(t)-A_1(t)w^{\frac{\alpha+1}{\alpha}}(t),t\geqslant t_{10}。\tag{5}$$

2) 假设 $\beta<\alpha$,定义函数 $w(t)=\dfrac{\rho(t)r(t)(z''(t))^{\alpha}}{(z'(t))^{\beta}}$。于是有 $w(t)>0$,且:

$$w'(t)\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\rho(t)Q(t)z^{\beta}(\sigma_1(t))}{(z'(t))^{\beta}}-\frac{\beta(z''(t))^{1-\frac{\alpha}{\beta}}}{(r(t)\rho(t))^{\frac{1}{\beta}}}\frac{(\rho(t))^{1+\frac{1}{\beta}}(r(t))^{1+\frac{1}{\beta}}((z''(t))^{\alpha})^{1+\frac{1}{\beta}}}{((z'(t))^{\beta})^{1+\frac{1}{\beta}}}=$$

$$\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)\left(\frac{z(\sigma_1(t)}{z'(t)}\right)^{\beta}-\frac{1}{(z''(t))^{\frac{\alpha-\beta}{\beta}}}\cdot\frac{\beta}{(r(t)\rho(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t)。$$

从方程(1)和引理 1 的情形(Ⅱ)知 $z'''(t)\leqslant0,t\geqslant t_8$,则 $z''(t)$是一个减函数。

又因为 $z''(t)>0,\dfrac{\alpha-\beta}{\beta}>0$,于是$(z''(t))^{\frac{\alpha-\beta}{\beta}}\leqslant(z''(t_8))^{\frac{\alpha-\beta}{\beta}}=\dfrac{1}{M_2},t\geqslant t_8$。其中:$M_2=(z''(t_8))^{-\frac{\alpha-\beta}{\beta}}$。

类似(4)式的推导,$\forall\, t\geqslant t_{10}$,有:

$$\left(\frac{z(\sigma_1(t))}{z'(t)}\right)^{\beta}=\left(\frac{z(\sigma_1(t))}{z'(\sigma_1(t))}\cdot\frac{z'(\sigma_1(t))}{z'(t)}\right)^{\beta}\geqslant\left(\frac{\int_{t_{10}}^{\sigma_1(t)}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u}{\int_{t_9}^{\sigma_1(t)}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\cdot\frac{\int_{t_9}^{\sigma_1(t)}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\beta}=K_2(t)。$$

其中:$K_2(t)=\left(\dfrac{\int_{t_{10}}^{\sigma_1(t)}\int_{t_9}^{u}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\mathrm{d}u}{\int_{t_9}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\beta}$,有:

$$w'(t)\leqslant\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)K_2(t)-\frac{\beta M_2}{(r(t)\rho(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t)=$$

$$-\rho(t)Q_2(t)+\frac{\rho'(t)}{\rho(t)}w(t)-A_2(t)w^{\frac{\beta+1}{\beta}}(t),t\geqslant t_{10}。$$

其中:$Q_2(t)=Q(t)K_2(t),A_2(t)=\dfrac{\beta M_2}{(r(t)\rho(t))^{\frac{1}{\beta}}}$。于是得到:

$$\rho(t)Q_2(t)\leqslant-w'(t)+\frac{\rho'(t)}{\rho(t)}w(t)-A_2(t)w^{\frac{\beta+1}{\beta}}(t),t\geqslant t_{10}。\tag{6}$$

令 $\overline{Q}(t)=\min\{Q_1(t),Q_2(t)\},A(t)=\min\{A_1(t),A_2(t)\},\lambda=\min\{\alpha,\beta\}=\begin{cases}\alpha,\beta\geqslant\alpha\\ \beta,\beta<\alpha\end{cases}$,则可将(5),(6)式合并起来写成:

$$\rho(t)\overline{Q}(t)\leqslant-w'(t)+\frac{\rho'(t)}{\rho(t)}w(t)-A(t)w^{\frac{\lambda+1}{\lambda}}(t),t\geqslant t_{10}。\tag{7}$$

于是有:

$$\int_T^t G(t,s)\rho(s)\overline{Q}(s)\mathrm{d}s \leqslant \int_T^t G(t,s)\left[-w'(s)+\frac{\rho'(s)}{\rho(s)}w(s)-A(s)w^{\frac{\lambda+1}{\lambda}}(s)\right]\mathrm{d}s =$$

$$G(t,T)w(T)+\int_T^t\left(\left[\frac{\partial G(t,s)}{\partial s}+\frac{\rho'(s)}{\rho(s)}G(t,s)\right]w(s)-A(s)G(t,s)w^{\frac{\lambda+1}{\lambda}}(s)\right)\mathrm{d}s =$$

$$G(t,T)w(T)+\int_T^t[\,|-g(t,s)|G^{\frac{\lambda}{1+\lambda}}(t,s)w(s)-A(s)G(t,s)w^{\frac{\lambda+1}{\lambda}}(s)]\mathrm{d}s, t>T\geqslant t_{10}。$$

利用不等式[4] $Bu-Au^{\frac{\lambda+1}{\lambda}}\leqslant\frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\cdot\frac{B^{\lambda+1}}{A^\lambda}$,有:

$$\int_T^t G(t,s)\rho(s)\overline{Q}(s)\mathrm{d}s \leqslant G(t,T)w(T)+\int_T^t \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\frac{1}{A^\lambda(s)}|g(t,s)|^{\lambda+1}\mathrm{d}s,$$

得到 $\frac{1}{G(t,T)}\int_T^t\left(G(t,s)\rho(s)\overline{Q}(s)-\frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\frac{1}{A^\lambda(s)}|g(t,s)|^{\lambda+1}\right)\mathrm{d}s\leqslant w(T)$。此式与(3)式矛盾,因此,引理1的情形(Ⅱ)不存在。 证毕

定理2 假设 $x(t)$ 是方程(1)的最终正解,$z(t)$ 满足引理1的情形(Ⅲ),且:

$$\int_{t_0}^\infty\int_w^\infty\left(\frac{1}{r(v)}\int_v^\infty\int_c^d q(u,\xi)\mathrm{d}\xi\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}v\mathrm{d}w=\infty。\tag{8}$$

则 $\lim\limits_{t\to\infty}x(t)=0$。

证明 由引理1的情形(Ⅲ),$\exists c\geqslant 0$,使得 $\lim\limits_{t\to\infty}z(t)=c$。因此,断言 $c=0$。如果 $c>0$,则取 $t_{11}=\max\{t_5,t_7\}$,$\forall t\geqslant t_{11}$,有 $x(t)\geqslant z(t)\geqslant c$,且:

$$(r(t)(z''(t))^\alpha)'=-\int_c^d x^\beta(\sigma(t,\xi))q(t,\xi)\mathrm{d}\xi\leqslant -c^\beta\int_c^d q(t,\xi)\mathrm{d}\xi, t\geqslant t_{11}。$$

于是有:

$$\int_t^\infty(r(u)(z''(u))^\alpha)'\mathrm{d}u\leqslant -c^\beta\int_t^\infty\int_c^d q(u,\xi)\mathrm{d}\xi\mathrm{d}u, t\geqslant t_{11};$$

$$z''(t)\geqslant c^{\frac{\beta}{\alpha}}\left(\frac{1}{r(t)}\int_t^\infty\int_c^d q(u,\xi)\mathrm{d}\xi\mathrm{d}u\right)^{\frac{1}{\alpha}}, t\geqslant t_{11}。$$

将上式在$[t,\infty)$上积分,再在$[t_{11},\infty)$上积分,得到 $z(t_{11})\geqslant c^{\frac{\beta}{\alpha}}\int_{t_{11}}^\infty\int_w^\infty\left(\frac{1}{r(v)}\int_v^\infty\int_c^d q(u,\xi)\mathrm{d}\xi\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}v\mathrm{d}w$,与(8)式矛盾,于是有 $c=0$,即 $\lim\limits_{t\to\infty}z(t)=0$。

可以断言:$x(t)$ 有界。假若 $x(t)$ 无界,则存在一个序列 $\{t_n\}$,$\lim\limits_{n\to\infty}t_n=\infty$,使得 $\lim\limits_{n\to\infty}x(t_n)=\infty$,其中 $x(t_n)=\max\{x(u):t_0\leqslant u\leqslant t_n\}$。由条件($H_2$)知,$\lim\limits_{t\to\infty}\inf\tau(t,\xi)=\infty$,于是,对充分大的 n,$\tau(t_n,\xi)>t_0$。又由条件(H_2)知,$\tau(t,\xi)\leqslant t$,$t\geqslant t_0$,则:

$$x(\tau(t_n,\xi))=\max\{x(u):t_0\leqslant u\leqslant\tau(t_n,\xi)\}\leqslant\max\{x(u):t_0\leqslant u\leqslant t_n\}=x(t_n)。$$

于是,有:

$$z(t_n)=x(t_n)-\int_a^b p(t_n,\xi)x(\tau(t_n,\xi))\mathrm{d}\xi\geqslant x(t_n)-\int_a^b p(t_n,\xi)x(t_n)\mathrm{d}\xi\geqslant x(t_n)-p_0x(t_n)。$$

则 $\lim\limits_{n\to\infty}z(t_n)=\infty$,与 $\lim\limits_{t\to\infty}z(t)=0$ 矛盾,于是 $x(t)$ 有界。因此,可设 $\lim\limits_{t\to\infty}\sup x(t)=c_0$,其中 $0\leqslant c_0<\infty$,则存在一个序列 $\{t_i\}$,$\lim\limits_{i\to\infty}t_i=\infty$,使得 $\lim\limits_{i\to\infty}x(t_i)=c_0$。

若 $c_0>0$,取 $\varepsilon=\frac{c_0(1-p_0)}{3p_0}>0$,则有 $x(t_i)<c_0+\varepsilon$,故:

$$0=\lim_{i\to\infty}z(t_i)\geqslant\lim_{i\to\infty}[x(t_i)-p_0x(t_i)]\geqslant c_0-p_0(c_0+\varepsilon)=\frac{2c_0(1-p_0)}{3p_0}>0。$$

这是一个矛盾,于是 $c_0=0$,则 $\lim\limits_{t\to\infty}x(t)=0$。 证毕

定理3 假设(3)式和(8)式均满足,则方程(1)的每一个解都振动或收敛到 $0(t\to\infty)$。

证明　从定理 1 知,引理 1 的情形(Ⅱ)不存在。

假设引理 1 的情形(Ⅲ)满足,则由定理 2 知$\lim\limits_{t\to\infty}x(t)=0$。

如果引理 1 的情形(Ⅰ)满足,则$\lim\limits_{t\to\infty}z(t)=-\infty$。与定理 2 证明类似,可知 $x(t)$与 $x(t)$都有界,则与$\lim\limits_{t\to\infty}z(t)=-\infty$矛盾,故引理 1 的情形(Ⅰ)不会发生。

如果引理 1 的情形(Ⅳ)满足,则$\lim\limits_{t\to\infty}z(t)=c_1\leqslant 0$,与定理 2 的证明类似,可得$\lim\limits_{t\to\infty}x(t)=0$。　　证毕

例　考虑下面的三阶微分方程:

$$\left(x(t)-\frac{1}{2}\int_0^{\frac{\pi}{2}}x(t-\xi)\mathrm{d}\xi\right)'''+\int_{\pi}^{2\pi}x^3\left(\frac{t+\xi}{2}\right)\mathrm{d}\xi=0。\tag{9}$$

其中:$r(t)=1,\alpha=1,\beta=3,a=0,b=\frac{\pi}{2},c=\pi,d=2\pi,p(t,\xi)=\frac{1}{2},\tau(t,\xi)=t-\xi,\sigma(t,\xi)=\frac{t+\xi}{2},q(t,\xi)=1,\lambda=1$。

选取 $\rho(t)=1,G(t,s)=(t-s)^2,g(t,s)=2,t_9=t_{10}=t_0=0$,于是有:

$$A(t)=\min\{1,3M_2\}=A(\text{常数}),\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s=\int_{t_0}^{\infty}\mathrm{d}s=\infty,Q(t)=\pi,\overline{Q}(t)=\frac{\pi(t+\pi)^6}{(8t)^3},$$

$$\int_{t_0}^{\infty}\int_{w}^{\infty}\left(\frac{1}{r(u)}\int_{v}^{\infty}\int_{c}^{d}q(u,\xi)\mathrm{d}\xi\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}v\mathrm{d}w=\int_{t_0}^{\infty}\int_{w}^{\infty}\int_{v}^{\infty}\int_{c}^{d}\mathrm{d}\xi\mathrm{d}u\mathrm{d}v\mathrm{d}w=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{G(t,t_0)}\int_{t_0}^{t}\left(G(t,s)\rho(s)\overline{Q}(s)-\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}(A(s))^{\lambda}}\cdot|g(t,s)|^{\lambda+1}\right)\mathrm{d}s=$$

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_0^t\left[(t-s)^2\frac{\pi(s+\pi)^6}{(8s)^3}-\frac{1}{A}\right]\mathrm{d}s=\infty。$$

由定理 3 知方程(9)的解 $x(t)$振动或收敛到 $0(t\to\infty)$。

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Oscillation of Third Order Neutral Differential Equations with Distributed Delay

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Abstract: [**Purposes**] There are few results on the oscillation of solutions of third order neutral distributed delay differential equations. Most of them focus on the situation of $\alpha=\beta$. The objective is to study the oscillation of solutions for a class of third-order neutral distributed delay differential equations in the cases of $\beta\geqslant\alpha$ and $\beta<\alpha$. [**Methods**]The oscillation of solutions of third-order neutral distributed delay differential equations is studied by using generalized Riccati transformation technology and Yang inequality. [**Findings**] Some new sufficient condition for each solution of the equation to oscillate or converge to zero is established. [**Conclusions**]The results generalize and improve some well-known oscillation criteria in the literature. Finally, an example is given to illustrate the applicability of the new results.

Keywords: oscillation criteria; neutral differential equation; generalized Riccati transformation

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